2012

Inventory Systems with Transshipments and Quantity Discounts

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Inventory Systems with Transshipments and Quantity Discounts

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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2012

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December 11, 2012

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Abstract

Noble, Gregory D. Ph.D., Engineering Ph.D. program, Wright State University, 2012.
Inventory Systems with Transshipments and Quantity Discounts.

This research advances knowledge in the area of inventory systems and the relationship between competing retailers and suppliers of goods. This dissertation studies retailers that face uncertainty in the demand for goods and who purchase the goods from a single supplier. A game theoretic methodology is developed to analyze supplier pricing decisions and retailer quantity decisions and interactions in a competitive transshipment system where quantity discounts are offered. The system studied differs from previous work in the subject by introducing quantity discounts from supplier to retailer into a system of competitive retailers that transship stock. Analysis of these systems focuses on the identification of potential equilibrium actions by the retailers and how the supplier can influence the retailers through its choices. This dissertation identifies new criteria for stability within these systems. Understanding stable relationships is critical to the supplier's ability to influence the system to achieve coordinated, optimal performance. In a one supplier, two retailer transshipment system, all entities can achieve larger expected profits when the supplier offers the retailer a pricing contract with a two-block tariff quantity discount rather than a standard fixed pricing contract. Additionally, the supplier pricing decisions are examined. This analysis leads to a set of rules for setting the pricing contract parameters that maximize the supplier's expected profit. Three quantity discount

schemes are examined in the two retailer transshipment system: a two-block tariff, a two-part tariff, and an all-units discount. The two-block tariff discount results in the largest expected profit and greatest flexibility for the supplier. The topic is further researched through the examination of a three retailer system with transshipment and quantity discounts where there is a central retailer. The examination of this system shows the potential value to each entity of one of the retailers entering into a transshipment contract with another retailer.

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Chapter 1: Introduction

An important aspect of managing the logistics and level of customer service for a firm is making decisions on the inventory systems associated with the firm. The field of inventory theory which seeks to optimize the costs associated with management of inventory has a long history. It is essential for firms to implement effective inventory systems to ensure that items are delivered to customers when they are needed and in the quantities that are needed, while controlling the costs associated with acquiring and holding of inventory. Companies that can effectively manage inventory have a competitive advantage through potentially higher profits driven by reductions in inventory holding costs, reductions in required storage space, reductions in the quantity of lost sales and reductions in unsold inventory. An important and relatively new topic to the area of inventory theory is the role of supplier control and retailer decisions within a transshipment system.

The amount of supplier control in multilevel supplier/retailer systems varies from full control of all aspects, to control over little. Possible aspects that are controllable by the supplier include, but are not limited to: cost per unit, quantity discount schemes, retail price, transshipment costs, rules for transshipment of stock, profit sharing agreements, and distribution schedules. Retailers seek independence in decision making so that retail systems can be tailored to return more profits and reduce costs for retailers through such things as transshipment agreements. The level of control that suppliers and retailers have determines how each can affect their own and overall expected profits.

Consider this motivating scenario: two competing electronics retailers, Best Buy and Rex, sell the same flat panel television model from a common supplier, say Vizio. The retailers decide to cooperate in a limited way based on a transshipment agreement. The transshipment agreement, which the retailers enter into without direct influence from the supplier, allows the two retailers to share stock of the item. Due to a short product life cycle, shipments of this particular television are only delivered to the retailer locations from the supplier (Vizio) once. After the primary selling season for the television, the majority of the demand for the item has been realized. If one of the retailers, Best Buy for example, runs out of stock for the television then Best Buy would request items from Rex in order to fulfill Best Buy's unmet demand. Based on the transshipment agreement, as long as Rex has excess stock of the television after satisfying their own demand, Rex would send the requested units to Best Buy at an agreed-upon price. Thus Rex would transship units to Best Buy in this situation. By doing this, Best Buy avoids the loss of the sale. In the transshipment agreement, the cost to a retailer that receives a transshipped unit is typically higher than for a unit received directly from the supplier. Rex benefits from this agreement by selling off inventory that would otherwise sit unsold. Additionally, each retailer, Best Buy and Rex, may be able to order fewer units for safety stock because they can rely on each other to help meet variations in demand. Thus under the correct circumstances both retailers improve profitability through some combination of increased revenue and/or decreased inventory related costs. Figure 1 shows the flow of items in a multi-echelon inventory system with transshipments.

Another important feature of multi-echelon inventory systems in practice is the use of quantity-based discounts by suppliers to motivate larger orders from retailers. For example, *Vizio* could offer *Rex* a 10% discount on all units above 100 that *Rex* orders. The direct impact of this is that *Rex* can now potentially sell those units at a higher margin, or pass the discount along to their customers.

From the perspective of the supplier, quantity discounts serve as a method for increasing sales, though at a reduced cost per unit. From the perspective of the retailers, quantity discounts offer a tradeoff of risks. By choosing a larger ordering quantity to take advantage of a quantity discount offering, the retailers have a reduced risk of unmet customer demand, but an increased risk of excess inventory. Thus the retailer's decision of whether or not to take advantage of a quantity discount relies heavily on both the salvage value (or inventory holding costs in the case of multi-period models) and the penalty cost of missed sales. The supplier can use this information to offer a quantity discount scheme that will be beneficial to both the supplier and retailer.

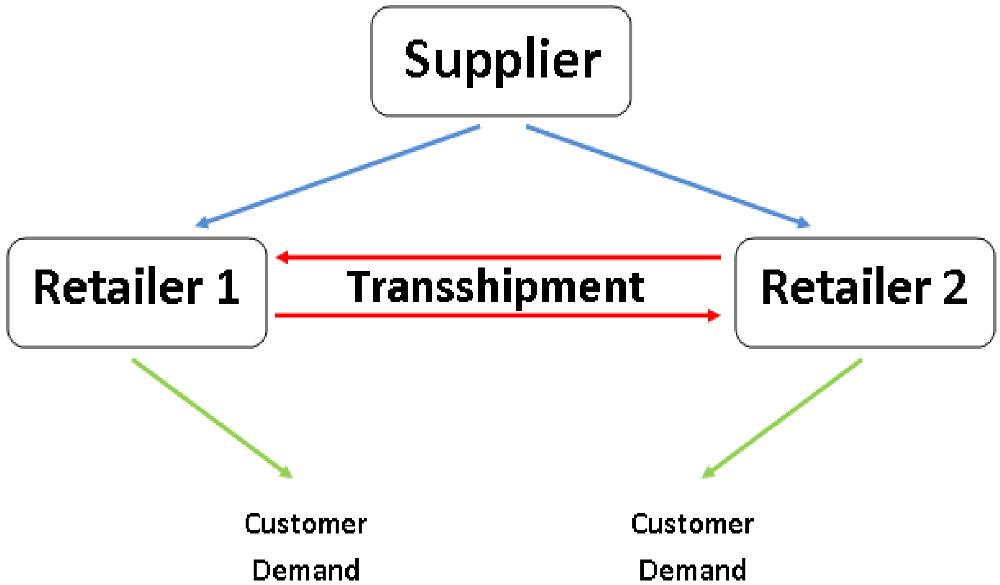

Figure 1: Flow Diagram Showing Transshipment

From the retailer's perspective transshipment allows for inventory sharing of a product between locations where there is sufficient demand for that product. The most common advantage identified in the inventory and supply chain literature from transshipment is that it allows for lower stock levels to be held at all locations, thus reducing inventory holding costs.

The interaction of incentives between quantity discounts and transshipment agreements has not been studied in the past and leads to significant new insights into the behavior of suppliers and retailers. This dissertation contributes to the existing literature by studying transshipment systems with quantity discounts involving competing retailers and a supplier using the quantity discount parameters to exert a coordinating influence on the system. A significant literature on transshipment systems has appeared within the past 15 years and includes many different strategies on the utilization of transshipment systems to improve system performance. Literature on quantity discounts in the various models for inventory control has a long history, but has not included its interactions with transshipment systems.

The recent work in the area of transshipment has shown that transshipment creates opportunity for improving profits by allowing for lower stock levels to be held by retailers while achieving the appropriate levels of service (e.g. (Lee, 2007), (Rudi, 2001)). The existing literature generally studies competing retailers with symmetric characteristics (costs, demands) as transshipment partners and uses the tools of game theory to show the existence of optimal equilibrium actions for the retailers where their ordering quantities are symmetric. In this work we show how quantity discounts, offered by a single supplier to either two or three independent retailers with symmetric characteristics, but working as transshipment partners, can lead to optimal equilibrium actions where their ordering quantities are not symmetric (i.e. unequal ordering quantities). Furthermore, the equilibrium identified in this situation can be more profitable to both the retailers and the supplier. The retailers naturally agree to this arrangement in an equilibrium that has non-symmetric ordering quantities for the retailers, yet achieves an equal division of profits between the retailers. The supplier's careful choice of the quantity discount scheme is able to achieve this equal division of profits without an explicit profit sharing agreement among the retailers. In these situations, the retailers choose to offer limited cooperation through the transshipment agreement, and the supplier fills a coordinating role that leads to improved performance for both retailers and the supplier.

The benefit for the retailers is improved profitability from the additional decision-making flexibility flowing from the transshipment relationships and in potentially achieving sales volumes that engage the quantity discount. The additional power of this analysis for the supplier is that with the identification of the multiple equilibrium actions that the retailers can take, as a function of the quantity discount parameters, the supplier can now consider how the quantity discount parameters create an opportunity for coordinating the behavior of competing retailers to benefit the supplier, the retailers, or the overall system. This can be accomplished without direct control of the ordering process of the retailers. This is particularly important because in these systems we assume that the retailers make choices independent of the direct control of the supplier, and thus it is advantageous for the supplier to understand the impact of optimization by each retailer and competition between the retailers so that the supplier can create incentives for improved performance.

The basis of the model for each retailer is the newsvendor model, which is a classic model in inventory theory that models the decision of an inventory manager facing an uncertain demand by trading off 1) the cost of inventory to help meet demand and 2) the cost of shortages from unsuccessfully fulfilling the demand. In the newsvendor model, the decision maker purchases stock at the beginning of a time period, can salvage it at the end of a time period, and can no longer sell it during the following time periods. The model of transshipments within a one-warehouse, two-retailer, multi-echelon inventory system is based on the transshipment model of (Rudi, 2001).

1.1: Overview of Chapter Two

Chapter two discusses the relevant background information including the areas of transshipments, quantity discounts, and game theory. The research discussed in this chapter lays the ground work for the research discussed in this dissertation. Most notably

(Rudi, 2001) provides the basis for the model developed in chapter three. (Rudi, 2001) compares both decentralized and centralized transshipment systems and makes comparisons between the two systems. It is evident from the recent literature that transshipment systems lead to lower stock levels for the retailers. Additionally, the combination of lower stock levels in conjunction with the possibility of retailers having their unmet demand met by a transshipment partner and the possibility of fulfilling another retailer's unmet demand with excess inventory leads to increased net profit. In the examples that were studied, increases in retailer profit. Though not always feasible, the transshipment systems can be further improved through central coordination of inventory ordering quantities rather than decentralized, competing retailers. Additionally, (Dolan, 1987) provides the formulations of various quantity discounts. These are the basis of the comparison of models found in chapter 5. The literature on the topic of quantity discounts also covers supplier pricing decisions, retailer ordering decisions, and the relationship between the supplier and retailers. In some cases, centralized ordering and distribution rather than decentralized ordering can be considered as an option when making ordering decisions. Chapter 2 also discusses game theory as it applies to this research. The concept of a Nash equilibrium is summarized, which is essential to the game theoretic approach taken to solve transshipment problems. Foresight of other players' reactions to one's own actions is a central component in analysis based on game theory. The amount of foresight that a player has not only influences the decisions that are made by that player, but also affects the end results as well (e.g. ordering quantities, profits, etc.).

1.2: Overview of Chapter Three

Chapter three discusses a competitive two retailer transshipment game with quantity discounts. This section covers the formulation of the model as well as an example with results. The expected profit model as well as the optimal response functions of the retailers and transshipment event graphs are given in this chapter. Figure 2 shows the transshipment event graph for the transshipment with quantity discounts model. A transshipment event graph labels regions in the retailers' demand space with the direction and quantity of transshipment. Transshipment event graphs are important as an aid to understanding the way that transshipment works within the model. The different regions in the event graph are determined by the transshipment rules as they relate to each retailer's demand and ordering quantity.

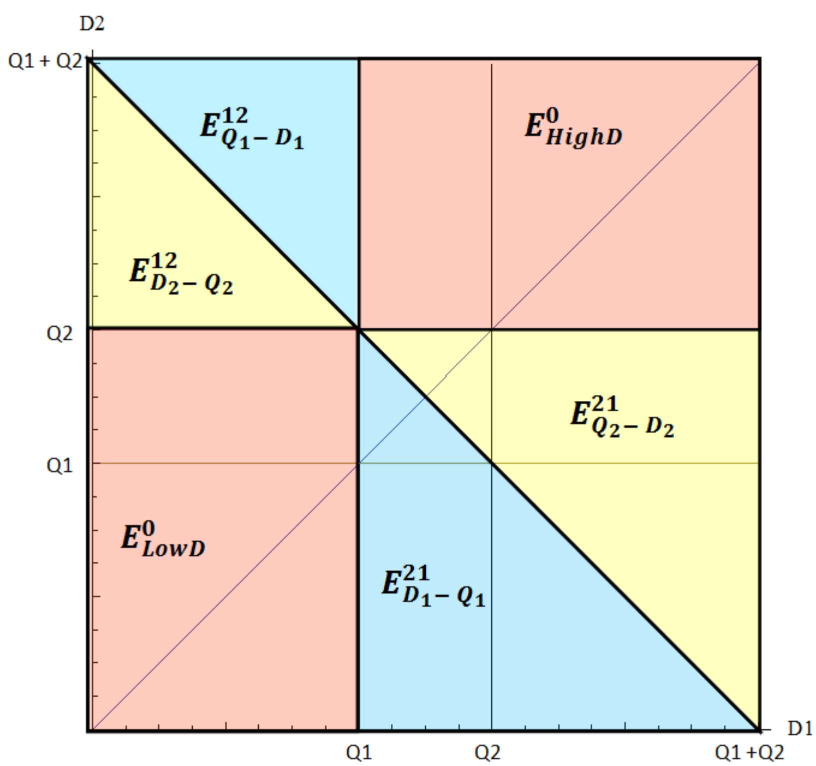

Figure 2: Graphical illustration of the transshipment events that occur in the transshipment with quantity discount model.

Optimization of the model leads to an analysis of optimal response functions for the retailers, focusing on the existence of equilibrium points as a function of the quantity discount parameters chosen by the supplier. An optimal response function is a function describing a retailer's optimal ordering quantity based upon the other retailer's ordering quantity. An equilibrium point is the intersection of two response functions, one for each retailer. Based on a derivative approach to profit maximization, when at an equilibrium, neither retailer has an incentive to change their quantities to improve their profit. An example of response functions and the resulting potential equilibria is shown in Figure 3.

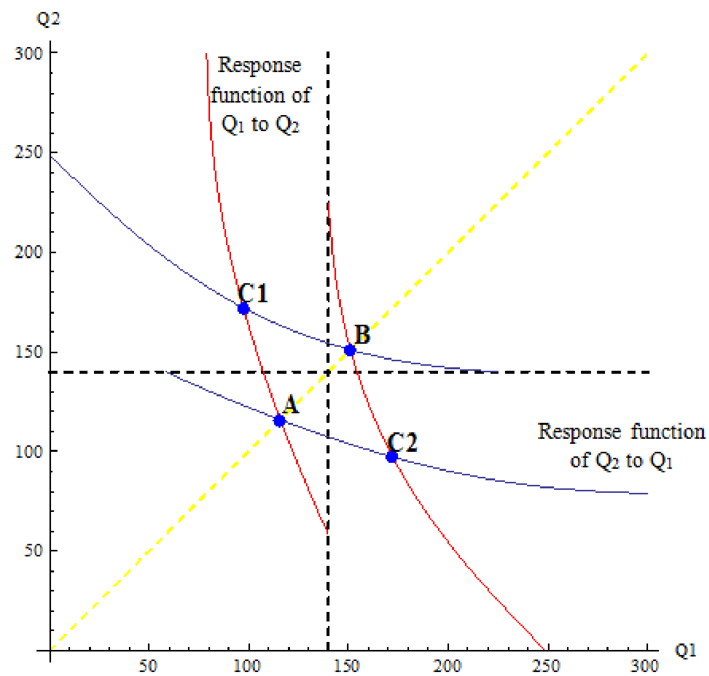

Figure 3: Plot of intersecting response functions showing the existence of four potential equilibrium points for two retailers with a discount triggering quantity at 140.

This chapter defines the requirements for a stable point and analyzes the expected supplier and retailer profits. The expected profits are used in conjunction with the retailers' response functions to determine the best supplier and retailer decisions. The supplier specifies the pricing contract through a normal cost per unit, reduced price per unit, and a discount triggering quantity. A stable point is a value of the discount triggering quantity where an equilibrium can be obtained on which all retailers will have their highest possible expected profit for that particular value of the discount triggering quantity. The identification of stable points in the analysis is essential in order to ensure that the retailers do not jockey to obtain a position of higher profit. This type of jockeying can lead to a suboptimal set of ordering quantities due to their competitive relationship.

Figure 4 shows an example of the retailers' expected profits as a function of the discount triggering quantity. Note that the existence of the equilibria is a function of the discount triggering quantity. Also, for the two retailer model, there is only one stable point when all of the equilibria are in existence. In Figure 4, this is the point at which the blue and yellow profit lines cross for the two retailers. From Figure 4 the supplier can gather information on the existence of the equilibria, retailer stability, and expected retailer profit. The supplier uses this information to choose pricing parameters that lead to stable solutions which result in higher profit for both the supplier and retailers.

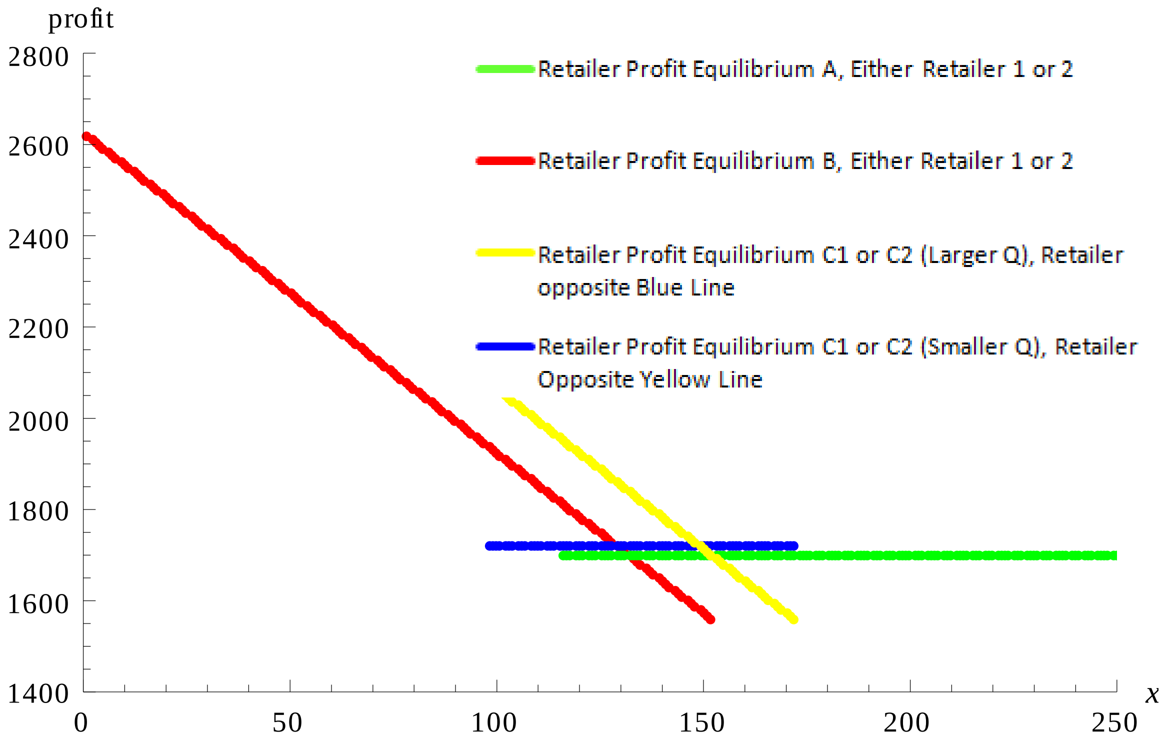

Figure 4: Retailer Profit for different values of the discount triggering quantity, comparing different ordering quantity combinations.

Through examples, the characteristics of retailer behavior are demonstrated with a study of expected profits from both the retailer's perspective and the supplier's perspective.

Using the two-block tariff quantity discount price structure in a transshipment model results in the two retailers having four potential equilibrium ordering quantity combinations to consider, rather than just one when the supplier does not offer a pricing contract with a quantity discount. Two of the potential retailer equilibrium combinations have symmetric retailer ordering quantities, while the other two potential retailer equilibrium combinations have non-symmetric ordering quantities. This is in spite of otherwise symmetric cost and demand parameters.

The quantity discount cost parameters are chosen by the supplier. Strategically, the new analysis shows that the supplier should choose the pricing parameters within a range so that the retailers have a non-symmetric equilibrium to consider, since this can lead to the highest profits for the supplier. Also, the supplier must carefully choose the quantity within the range that contains the non-symmetric equilibrium so that the retailers have equal profits, allowing the equilibrium to be stable, as defined below. When the supplier is successful in coordinating the retailer's actions, we find that the supplier's profit will be largest when the value of the non-discounted cost per unit from the supplier is as large as is feasible and the value of discounted cost per unit from the supplier is as small as is feasible. Decreasing the value of the discounted cost per unit from the supplier also increases both retailers' profits, while increasing the value of the non-discounted cost per unit from the supplier decreases both retailers' profits.

We further extend the understanding of equilibrium behavior in these systems. Due to the retailers' competitive relationship, they could jockey for position among the potential equilibriums so that they receive a higher profit for themselves and the other retailer receives a lower profit. A stable value of the discount triggering quantity is defined as a value of discount triggering quantity where the expected profit for one retailer is equal to the expected profit for the other retailer and the two retailers are at an equilibrium that has a larger expected profit for both retailers than any other equilibrium that is in existence for a particular value of the discount triggering quantity. Using this definition we find that a single value of the discount triggering quantity exists that results in the two symmetric retailers having different ordering quantities. Additionally, for this value of the discount triggering quantity, the expected profit for both retailers and the supplier is greater than that of the transshipment model without quantity discounts.

This research on a two retailer transshipment system with quantity discounts is significant to the current literature on transshipment because it shows that the optimal ordering quantities for the retailers have the following properties:

1 Satisfy the first order equilibrium conditions

2 Are not symmetric for the two retailers even though the quantity discount offered to the two retailers by the supplier is identical, all costs are symmetric for the two retailers, and the two retailers are competitive and thus not cooperating in making ordering quantity decisions.

3 Result in improved profits for each retailer over the symmetric equilibrium solutions.

4 Result in improved profits for the supplier.

1.3: Overview of Chapter Four

Chapter four discusses a system with three retailers and one supplier that offers a quantity discount. This system extends the two-retailer, one supplier system of Chapter three. One of the retailers serves as a "central retailer" which can transship with either of the other two "non-central" retailers. A graphical representation of this system is shown in Figure 5.

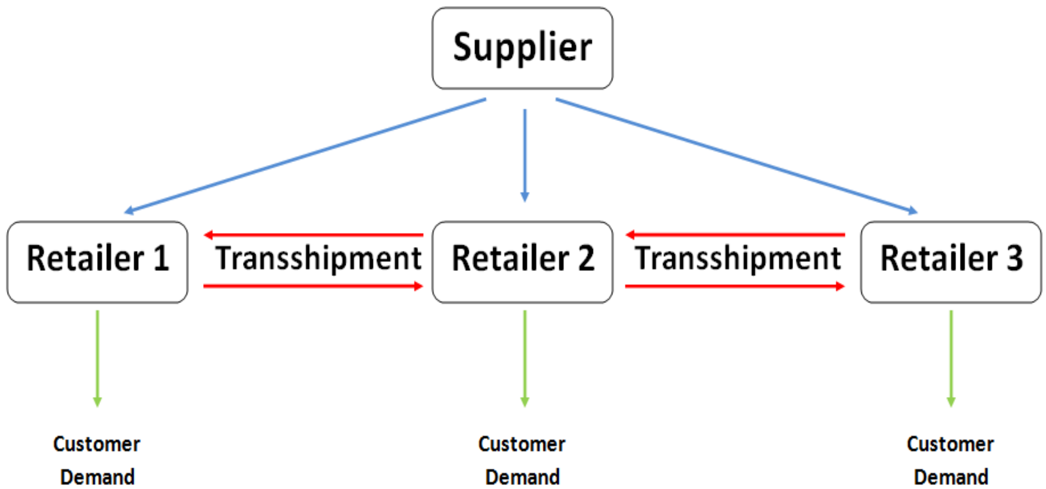

Figure 5: Flow diagram the the three retailer system with a central retailer.

The model for the three retailer system is first derived and verified. Event graphs, response functions, and example profit graphs are all provided in this chapter. The models for this system are much more complicated than the two retailer transshipment system. The complexity is mostly a result of the three dimensionality of the transshipment event space. Based on a study of the current literature, no work exists for such a system with three retailers with or without quantity discounts. Because this system shares many similarities with the two retailer system, direct comparisons can be made to determine the value to each entity within the system of one of the retailers entering into a transshipment agreement with another retailer. Figure 6 shows an example of a partial transshipment event graph for the three retailer system. The graph shows the space created from the three retailer demands and the regions within represent transshipment events. Like in the two retailer system, the event graphs here are important as an aid to understanding the way that transshipment works within the model. Since this model has not been seen previously in literature, the three-dimensional transshipment event graphs are very important here for verifying that the retailer expected profit model is correct. The different regions in the event graph are determined by the transshipment rules as they relate to each retailer's demand and ordering quantity. Regions differ from each other by the amount of inventory being transshipped and the direction of the transshipment.

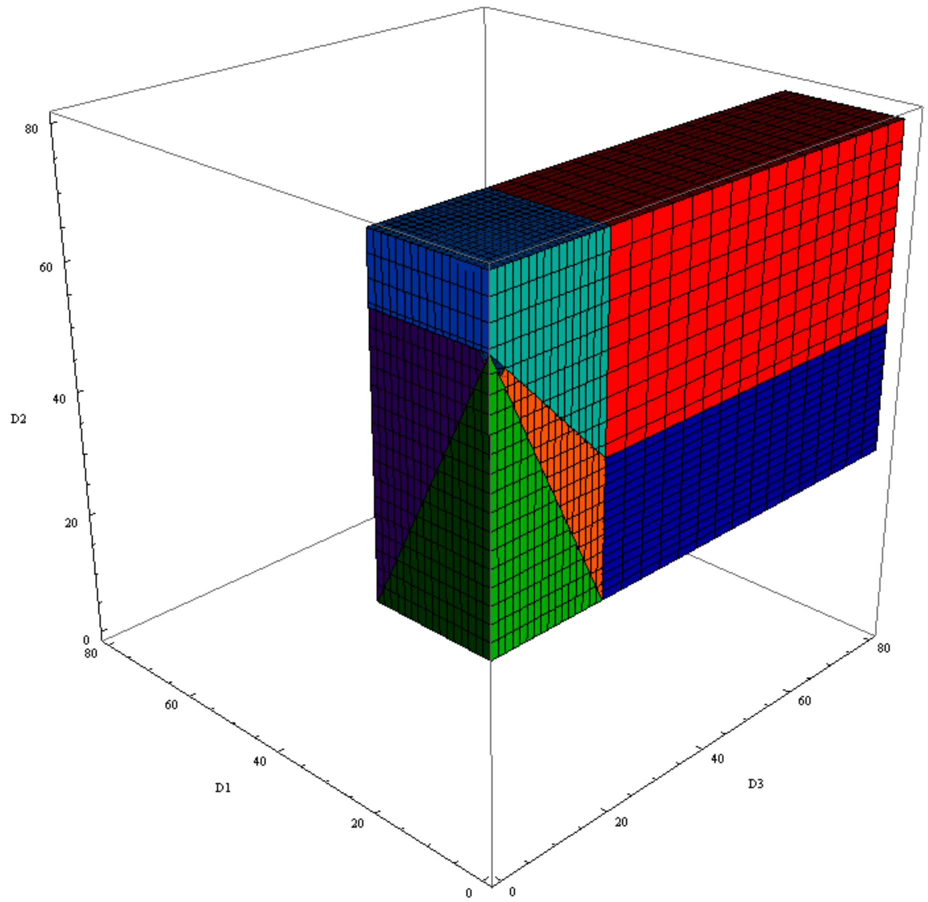

Figure 6: Example of a partial transshipment event graph for the three retailer system in the three dimensional demand space.

A game theoretic approach is used to determine the best supplier and retailer decisions within the three retailer system. Response functions for the three retailer system are found in order to determine the potential equilibria as shown in Figure 7.

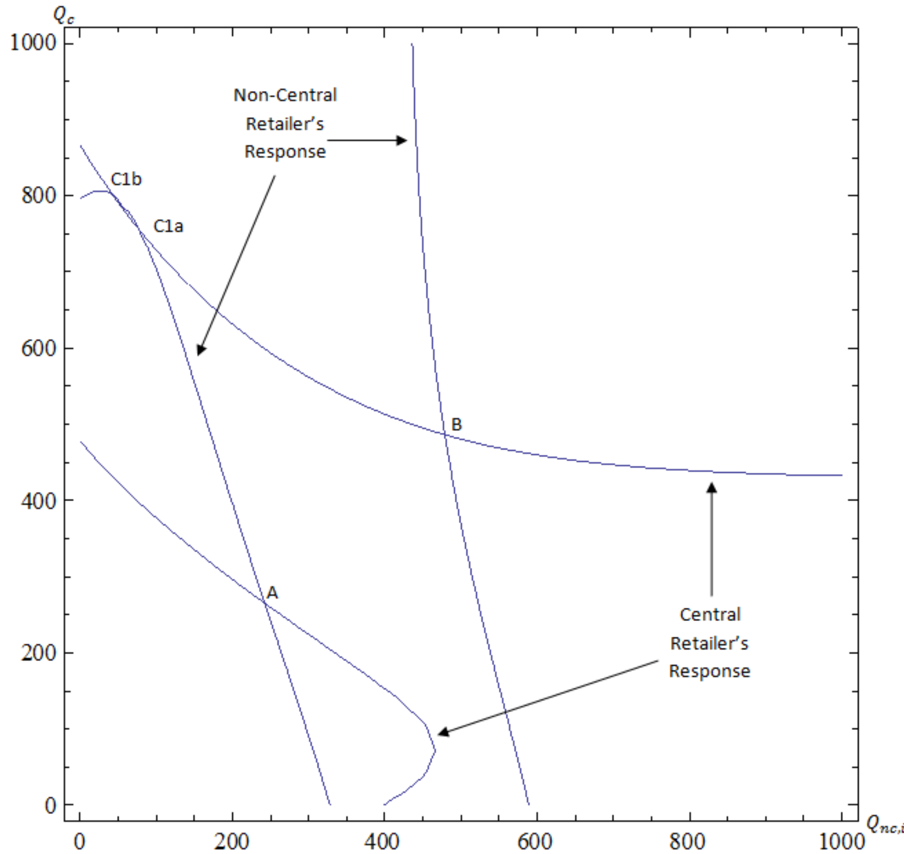

Figure 7: Example response functions for the three retailer system with potential equilibria A, B, C1a, and C1b.

The axes in Figure 7 represent the ordering quantities for the central retailer (vertical axis) and either of the two non-central retailers (horizontal axis). In this case, the non-central retailers have identical response functions. The central retailer's response function differs from the non-central retailers' response functions as seen in Figure 7. From the response functions, the range of existence for each of the equilibria can be determined and the expected profit graph for the retailers can be created as shown in Figure 8. Identifying stable equilibrium points requires careful analysis using Figure 8 over different regions of the discount triggering quantity.

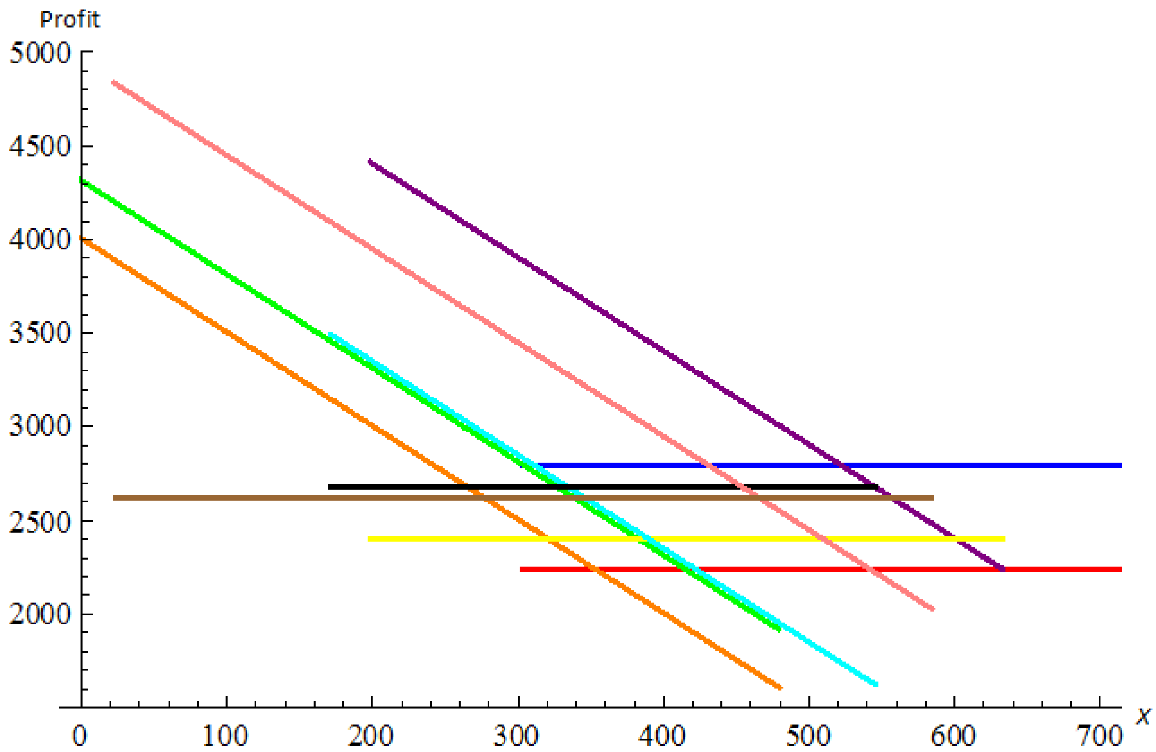

Figure 8: Example graph of the retailers' expected profit for the three retailer system as a function of the discount triggering quantity.

The research presented in this chapter also serves to determine the benefits and disadvantages to this system compared to the two retailer system. Through analyzing this system with a game theoretic approach, for the cases that were studied, the following changes occur to the expected profits:

1 Two retailers currently in a transshipment agreement have a decrease in expected profit if either of them takes on another transshipment partner (going from two to three retailers transshipping).

2 A retailer who previously operated independently as a standard newsvendor where quantity discounts are offered sees an increase in expected profit by entering in the transshipment agreement with the central retailer.

3 The central retailer has a greater expected profit than each of the two non-central retailers.

4 The supplier has an increase in expected profit over the two retailer system.

Consider two retailers that already have a two way transshipment agreement. If a third retailer joins into an transshipment agreement with one of the two original retailers, then both of the retailers currently in a transshipment agreement will have a decrease in expected profit. When the choice to enter into an additional transshipment contract belongs to the retailers and not the supplier, neither of two retailers that currently have a transshipment agreement should enter into a transshipment agreement with another retailer based on the parameters we studied in chapter four. If however one of them does enter into transshipment agreement with another retailer, then the retailer that was not originally part of the transshipment agreement and the supplier would have an increase in expected profit. Thus, the supplier and the independent retailer should be in favor of the additional transshipment agreement as both of them will experience an increase in expected profit.

1.4: Overview of Chapter Five

In chapter five three different quantity discount types are compared to determine how the choice of discount type influences the behavior of retailers, suppliers and their potential profit. The quantity discount types are evaluated in the two retailer transshipment with quantity discounts system from chapter three.

Three types of quantity discounts are compared in this chapter: a two-block tariff quantity discount, a modified two-part tariff quantity discount, and an all-units quantity discount. In a two-block tariff quantity discount a normal price is charged per unit purchased below a discount triggering quantity. Units purchased above the discount triggering quantity are charged a discounted price per unit. In a two-part tariff quantity discount a fixed cost is charged for the right to purchase any number of units at a uniform price per unit. In an all-units quantity discount if a discount triggering quantity is exceeded, a discounted price per unit is applied to all units. If the discount triggering quantity is not exceeded then the normal cost per unit is charged. Figure 9 shows a side by side comparison of the three quantity discount types focusing on the the total cost charged by the supplier to the retailer as a function of the quantity ordered. When compared in the two retailer transshipment system, chapter five demonstrates that the two-block tariff quantity discount is the best choice for the supplier. The two-block tariff quantity discount is a better option for the supplier than the two-part tariff quantity discount. This is due to both having equal profits for equilibria that exist when either quantity discount scheme is used and the two-block tariff quantity discount having more potential equilibria available. The two-block tariff quantity discount is better than the all-units quantity discount due to the more potential equilibria available to the two-block tariff quantity discount and a expected supplier profit that is at least as good as the all-units quantity discount for potential equilibria that both have in common. This result is based on discount triggering quantity flexibility (due to retailer stability) and expected supplier profit.

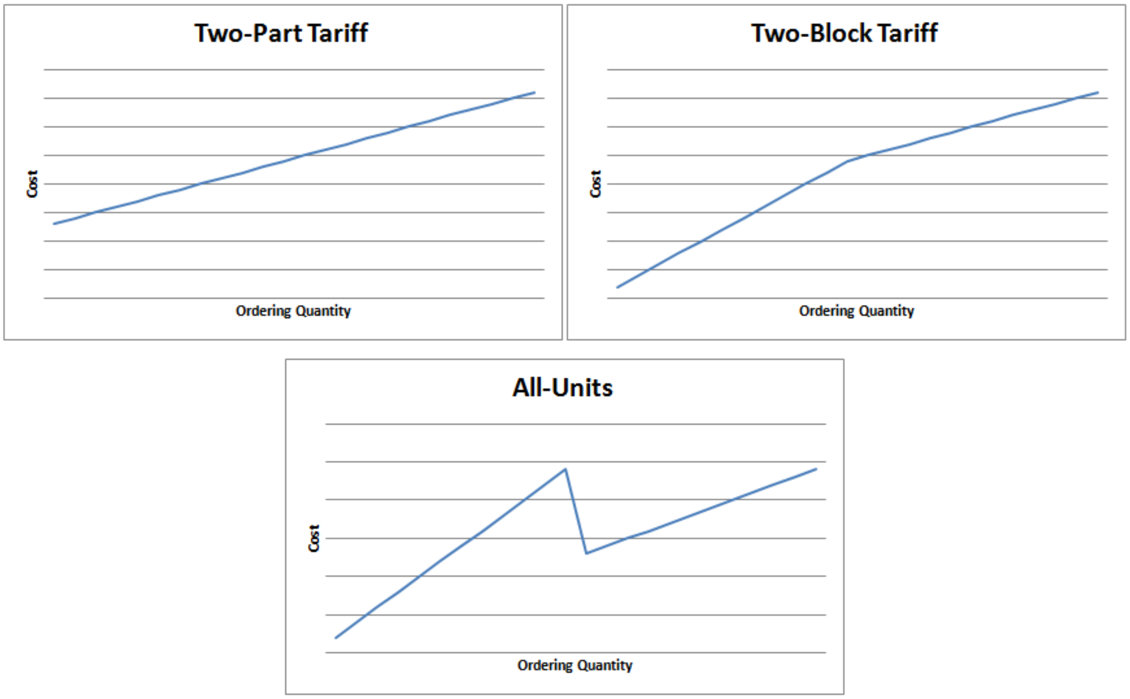

Figure 9: Comparison of the three quantity discount types for the total cost charged as a function of the quantity ordered.

A comparison between the two-block tariff quantity discount and a modified two-part tariff quantity discount shows that their response functions have similar components. How these components are composed into the complete response function differentiates the two approaches. For the two-block tariff quantity discount there is a discontinuity in the response function at the discount triggering quantity. The two-part tariff quantity discount does not have this discontinuity in its response function. Aside from this difference, the response functions are the same; resulting in the same potential equilibria. Using the stability criteria developed in chapter two, the two-block tariff quantity discount has four potential equilibria (A, B, C1, and C2, shown in Figure 3) for the retailers to choose from. When the stability criteria are applied to the two-part tariff quantity discount model the retailers can choose from only two potential equilibria (C1 and C2). Though the expected profits are the same for the C1 and C2 equilibria between both types, the two-block tariff quantity discount is preferred by the supplier due to the flexibility in stable regions and larger amount of choices for the discount triggering quantity over the two-part tariff quantity discount.

When comparing the two-block tariff quantity discount to the all-units quantity discount chapter five shows that the two-block tariff quantity discount is preferred over the all-units quantity discount because of increased expected profits and flexibility. The all-units quantity discount results in only two potential equilibria (A and B), while the two-block tariff quantity discount results in the existence of all four potential equilibria (A, B, C1, and C2) and thus is more flexible than the all-units quantity discount. The analysis in chapter five proves that when the retailers order according to Equilibrium B, the expected supplier profit for the two-block tariff quantity discount case is strictly greater than for the all-units quantity. The only time that the expected supplier profit between the two cases is equal is when the two retailers order according to Equilibrium A. When the retailers order according to any other equilibrium, the two-block tariff quantity discount either has a greater expected profit for the supplier or the equilibria do not exist in the all-units quantity discount case, in which case the two-block tariff quantity discount has the possibility of their existence.

Chapter six summarizes the main results and discusses possible future work.

Chapter 2: Literature Review

This chapter includes background information on three areas relevant to this research: transshipment, quantity discounts, and game theory. The section on transshipment explains the process of transshipment and summarizes recent research that is relevant to the topics in this dissertation. All the references described here assume that the retailers have independent demands where customers do not switch between retailers, unless it is specifically noted. The section on quantity discounts gives an overview of the different types of quantity discount models focusing on the more recent work that is relevant to the research presented in this proposal. The final section presents relevant topics in the area of game theory.

2.1: Transshipment

(Paterson, 2011) constructs a review on inventory models with lateral transshipments. The review of literature identifies articles by various characteristics of the system being evaluated such as the number of echelons and symmetry between the retailers. Articles are also identified by the type of ordering policies and transshipment characteristics used.

According to (Hanany, 2009) transshipments are the movement of stock in the same echelon level. In a transshipment system with two retailers, one will transship excess inventory to the other when it is needed based on customer demand. Assuming that it is cost advantageous for the two retailers to do so, once demand is realized, if there is both a retailer with excess stock and a retailer with excess demand, the retailer with excess stock will transship up to enough stock to meet the other retailer's excess demand in order to avoid taking a loss for the excess inventory. This also benefits the retailer with excess demand because the transshipment allows the retailer's demand to be met when the other retailer has excess stock. For example, assume that retailer 1 and retailer 2 each have 60 units of stock available. Also assume that the realized demand is 50 units for retailer 1 and 65 units for retailer 2. Retailer 1 can improve the level of service to retailer 2 customers by transshipping 5 units to retailer 2. This also benefits retailer 1 because excess stock not needed to serve retailer 1 customers would be sold to retailer 2. Retailer 2 benefits by receiving immediately needed stock from retailer 1 to meet the previously unmet 5 units of demand, though possibly at a higher unit price than if sourced directly from the supplier. Additionally, when this additional source of transshipped supply is considered, both retailers benefit by being able to reduce their ordering quantities from the product supplier and earn higher average profits than they would without using transshipments (Rudi, 2001).

(Rudi, 2001) shows that when transshipment is used, retailer orders at each location may not be optimal for the retailer, depending on how the performance of the system is being optimized. More specifically, a comparison is made between systems where local decision makers optimize their own performance and systems where a single central decision maker optimizes the overall performance. In this model high transshipment prices (the amount received by a retailer to transship its excess inventory) make it profitable for each firm to carry more inventory, while lower transshipment prices make it more profitable for each firm to rely on the other firms for transshipments to handle high demand. A newsvendor model is used to determine the best inventory levels to get the maximum average profit per firm. (Rudi, 2001) shows that the average profit per firm is higher when using central coordination than when the local decision makers seek to optimize their own profits.

In (Xiao, 2008) a transshipment game between two retailers with seasonal products is examined. The problem examined in (Xiao, ,2008) is similar to the problem examined in (Rudi, 2001) with a few differences which include the option to reject a transshipment request and the possibility that unmet demand will attempt to be met by the other retailer. Similar to (Rudi, 2001) they compare a decentralized transshipment system, a centralized transshipment system and a system without transshipment. Their results indicate that the centralized system has the highest joint revenue and that the decentralized system underperforms the centralized system in total joint revenue but outperforms the system without transshipment. (Xiao, 2008) concludes through numerical experiments that although transshipment can increase revenue for both retailers in some cases. In other cases the competition between the retailers can result in a loss in the joint revenue.

In (Zhao, 2009) an analysis of transshipment between two competing retailers is carried out. Two games are studied; a transshipment game and a substitution game. For each game the existence of a general pure-strategy Nash equilibrium is established for retail and safety stocks. The model follows closely to a single period newsvendor model. Decisions in this scenario include the price at which to transship, the inventory kept, and when to transship inventory to a competing retailer. After an analysis, it was found that the models are mainly dependent on transshipment price and the chance that a sale is lost. It is also found that transshipment is less attractive as the degree of competition increases, which is measured through the likelihood and cost of customers switching from one retailer to another. They also found that transshipment between retailers never leads to a lower retail price and never leads to a higher level of safety stock, so there is no benefit to customers.

In (Zou, 2010), an analytical model is developed in order to examine a system where there is transshipment between two competing firms. They find that transshipment systems might not be cost effective for firms if the firms are operating in an environment where customers are likely to switch from a firm that is experiencing a stockout to a firm with available inventory. In such environments, customers are less loyal to the firms and the firms are more competitive towards each other which results in a higher customer switching rate. The authors find that in these cases it is more beneficial to the firms to allow the customers to switch to another firm instead of utilizing transshipment. However, they do find that it is beneficial for the firms to use transshipment when there is high customer loyalty. Additionally they find that when both firms are identical in market demand and cost parameters that there exists a unique transshipment price that is optimal for both firms. In contrast, when the two firms are not identical in market demand and cost parameters, the firm which has a lower penalty cost or a smaller demand variance will prefer a lower transshipment price and receive more benefits than the other firm from transshipment.

In (Wong, 2005), a system where a company shares its spare parts for repairs is explored. A distinction is made between lateral transshipments and delayed lateral transshipments, where lateral transshipments occur when there are no backorders in the system and it is triggered by an item failure, where delayed lateral transshipments occur when the system has backorders and it is triggered by a repair completion. They formulate a multi-dimensional Markovian Model examining various combinations with or without delayed lateral transshipment and zero or non-zero lateral transshipment. It is shown that through the use of delayed lateral transshipments, the expected number of backorders can be significantly reduced. This (Wong, 2005) also notes that the assumption of instantaneous lateral transshipment can lead to non-optimal inventory stocking decisions, especially in cases where lateral transshipments do not occur very fast and, in cases for repairs, when the cost of unit-down time is high.

The focus of (Zhao, 2006) is an inventory transshipment problem in a decentralized dealer network. A game theory approach is used to determine inventory strategies of the individual dealers. Dynamic programming is used to prove that the optimal policy for an individual dealer is a stationary (S, K, Z) policy, where S is the base-stock level, K is the threshold rationing level (when a dealer should fill a request), and Z is the requesting level (when a dealer should send out a request). It is found that high-volume dealers are more sensitive to the cost of transshipment in their choice of the stocking and rationing levels than low-volume dealers, whereas low-volume dealers are more sensitive to the cost of transshipment in their choice of the requesting level. Additionally it is found that when the incentive for transshipment increases (manufacturer incentives, etc.), in a small dealer network, dealers decrease their rationing and stocking levels, whereas in a large dealer network, dealers respond by decreasing their rationing levels and increasing their stocking levels.

(Herer, 2001) defines the dynamic transshipment problem. (Herer, 2001) analyzes a system where transshipments are used between two retailers in a dynamic deterministic demand environment over a finite planning horizon. The goal is to determine the amount of stock to order from a single supplier and the amount of inventory to transship each time period while minimizing costs. A polynomial time algorithm is developed to determine the optimal strategy for the dynamic transshipment problem for the two retailers over a finite planning horizon. The dynamic transshipment problem is further examined in (Herer, 2003) through the addition of multiple retailers. In (Herer, 2003) a multi-location supply chain where transshipments are allowed is analyzed. The targeted scenario by (Herer, 2003) is a dynamic deterministic demand environment where each location has known demand for each period and the locations are centrally controlled. It is shown that problem is NP-hard. An optimal algorithm is developed in order to solve instances of the problem with fewer retailers. Additionally, a heuristic algorithm is developed in order to solve instances where the number of retailers is large.

(Lien 2011) Introduces chain configurations in transshipment systems where each entity is linked through a single connected loop. It is shown that a chain configuration has advantages over the configurations suggested in literature. In addition they demonstrate the efficiency and robustness of chain configurations for more general scenarios and provide managerial insights regarding preferred configurations for different problem parameters.

(Shao 2011) examines transshipment incentives in a decentralized supply chain where a single supplier is available to distribute product to independent retailers. It is shown that the manufacturer prefers to set the transshipment price as high as possible, whereas retailers prefer a lower transshipment price. Transshipment incentives in a system where the retailers are under joint ownership are also examined because the transshipment price does not play a role. It is found that when the decentralized retailers control the transshipment price, a lower price is chosen, which results in higher profits for the retailers and lower profit for the supplier. Thus, the supplier prefers to work with retailers under a joint ownership rather than independent retailers.

(Hu, 2007) examines a two-location production/inventory model where each location makes production decisions and is subject to uncertain capacity. (Hu, 2007) seeks to find the existence of a set of coordinating transshipment prices that induce the local decision makers to make inventory and transshipment decisions that are globally optimal. Sufficient and necessary conditions for the existence of a unique pair of coordinating transshipment prices are derived. It is shown that coordinating prices may exist for only a narrow range of problem parameters and (Hu, 2007) explores conditions for when this can happen. Additionally, a study is performed on the effects of demand and capacity variability on the magnitude of coordinating transshipment prices.

(Wanke, 2009) presents a framework for deciding whether and how inventories should be pooled, using the consolidation effect as a cornerstone tool to measure inventory costs, service levels, and total costs. Sensitivity analyses on mathematical expressions are performed to determine when one alternative is preferable in terms of total costs. Real settings are also presented for the framework developed.

(Olsson 2009) examines a single-echelon inventory system with two identical locations. It is assumed that each location applies a (R, Q) policy and shows that the optimal policies are not necessarily symmetric even though the locations are identical. The optimal policies are chosen based on the minimizing the joint costs from the two retailers. (Olsson, 2010) examines a similar system with the main difference being unilateral transshipment.

(Huang, 2010) studies a newsvendor game with transshipments, where *n* retailers face a stochastic demand for an identical product. Two methods for distribution of residual profit are compared; transshipment prices and dual allocations. Transshipment prices are selected before the demand is known; dual allocations are obtained by calculating the dual prices for the transshipment problem and are calculated after observing the true demand. It is shown that neither allocation method dominates the other. For *n* retailers they recommend a transshipment price agreement where a neutral central depot is used to coordinate the transshipments over a dual allocation agreement because of the ease of implementation.

(Seifert, 2006) compares integrated and dedicated supply chains. (Seifert, 2006) develops and solves mathematical models for dedicated and integrated supply chains. It is shown that the cost savings can be significant and that both retailers and customers benefit from an integrated supply chain. An analysis is also performed on how the optimal solutions depend on the characteristics of the supply chain and identify conditions under which it would be optimal to operate the virtual store without dedicating any inventory to the virtual store.

(Comez, 2012) studies a decentralized system of competing retailers. Here retailers have the option of rejecting a transshipment request. Each retailer decides on the initial order quantity and on the acceptance/rejection of each transshipment request. For two retailers it is shown that the retailers' optimal transshipment polices are dynamic and characterized by chronologically nonincreasing inventory holdback levels. The sensitivity of holdback levels is analytically studied and it is found that smaller retailers and geographically distant retailers benefit more from transshipments.

(Dong, 2004) studies the affects of both exogenous and endogenous wholesale prices on retailers and suppliers. When an exogenous wholesale price is used, the supplier is a price taker and cannot affect the wholesale price. When an endogenous wholesale price is used, the supplier can set the wholesale price. It is found that the supplier benefits from retailers' transshipments by charging a higher wholesale price, while retailers are worse off.

(Lee, 2007) proposes a new lateral transshipment policy, called service level adjustment. This policy differs from previous policies by integrating emergency lateral transshipment with preventative lateral transshipment to efficiently respond to customer demands. Emergency lateral transshipment mandates emergency redistribution from a retailer with ample stock to a retailer that has reached stockout. Preventive lateral transshipment reduces risk by redistributing stock between retailers that anticipate stockout before the realization of customer demands. Additionally, the proposed policy considers the service level to decide the quantity for lateral transshipment.

2.2: Quantity Discounts

A quantity discount is a pricing discount offered by a supplier to a customer for units above a specified purchase quantity. From a supplier's perspective, quantity discounts offer a means to entice retailers to purchase more units than they normally would. The supplier gains the advantage of selling more units but may not necessarily make more profit due to selling the units at a cheaper price. From a retailer's (customer's) perspective, quantity discounts offer a way to purchase units for a cheaper price; however, the supplier usually requires the retailer to purchase more units than the retailer would normally purchase in order to receive units at a discounted price.

Three types of quantity discounts are described by (Dolan, 1987). The first of the quantity discounts described by (Dolan, 1987) is a *two-part tariff*. When a two-part tariff is in effect the customer pays a fixed cost F for the right to purchase any amount of goods at a uniform price p. The total charge to the retailer for quantity q, $R(q)$ in a two-part tariff is:

$$R(q) = \begin{cases} F + pq, & q > 0, \\ 0, & q = 0, \end{cases} \text{ where } F > 0.$$

Another quantity discount, a *two-block tariff*, is a quantity discount in which a price p_1 is charged per unit for all units up to a quantity x and all units purchased greater than x are charged a price p_2 per unit (where $p_1 > p_2$).

$$R(q) = \begin{cases} p_1 q, & 0 \leq q \leq x \\ p_1 x + p_2(q - x) & \text{for } q > x. \end{cases}$$

The third quantity discount model described by (Dolan, 1987) is an *All-units quantity discount*. In an All-units quantity discount model once a quantity level x is exceeded, a discounted price is applied to all units such that:

$$R(q) = \begin{cases} p_1 q & \text{if } 0 \leq q \leq x, \\ p_2 q & \text{if } \quad q \geq x, \end{cases} \qquad \text{where } p_1 > p_2.$$

In this dissertation a two-block tariff quantity discount is used.

(Burnetas, 2007) examines how a supplier can use quantity discounts to influence the stocking decisions in a newsvendor system. Two different quantity discount structures are explored; an all-unit discount and two-block tariff quantity discount. Assumptions are made regarding the amount of information that the supplier knows about the buyer's demand. It is assumed that the supplier only knows the possible demand distributions of the buyer, and that the exact demand distribution is known to the buyer. Three different methods for the supplier to offer buyers; fixed packages, all-units quantity discount and two-block tariff quantity discounts, are compared. In this setting, an all-unit discount is always better than the two-block tariff quantity discount from the perspective of the supplier. Additionally, a fixed package pricing system results in the supplier earning larger profits than when using the all-unit discount or the two-block tariff quantity discount.

(Anand, 2008) uses a dynamic model to show that strategic inventories play a role in vertical contracts. A two-part tariff quantity discount is used as an example where in each period the buyers have the option of paying the fixed cost plus a cost per unit or purchase zero units at no cost. It is shown that the supplier pricing decisions are a function of the buyer's holding cost, such that when the buyer's holding costs are high, inventories are not held and thus do not affect the buyer and supplier decisions; however, when the holding cost is lower, inventories can play a role and affect buyer and supplier decisions.

(Munson, 2010) provides methodologies to calculate optimal ordering quantities for four different strategic purchasing configurations when either an all-units quantity discount or an incremental quantity discount is used. The different configurations include: complete decentralization, centralized pricing with decentralized purchasing, centralized purchasing with local distribution, and centralized purchasing with warehousing. It is shown through numerical example that the purchasing configuration can have a significant impact on costs. It is also shown through numerical example that under certain conditions, an incremental quantity discount schedule may reduce the incentive to order larger quantities than the all-units schedule.

(Shin, 2010) shows that the magnitude of optimal discounts that are scheduled by deterministic quantity discount models are not always large enough to cover the buyer's additional inventory stocking risks. A model is proposed that allows the supplier to identify a discount level that shares the buyer's increased risk that is associated with a temporary overstocking and ensures that both the buyer and the supplier benefit from the model. The proposed model includes a flexible supplier discount offer that is dependent on the magnitude of the buyer's demand variability.

(Kokangul, 2009) uses an integrated analytical hierarchy process and mathematical programming to solve the problem of supplier selection when quantity discounts are present. Mathematical programming is used in order to maximize the total value of purchase, minimize the total cost of purchase, and simultaneously maximize the total value of purchase and minimize the total cost of purchase. The analytical hierarchy process uses a pairwise comparison to calculate a weight for each of the suppliers which are then assigned as coefficients in the objective function to determine the optimal order quantities from each of the available suppliers.

(Chang, 2006) proposes a mixed integer formulation is for a single item multi-supplier system with variable lead-time, price-quantity discount, and resource constraints. (Chang, 2006) claims that the global optimal solution from the proposed model is better than the local optimal solution obtained by heuristic procedures in traditional methods. Examples are used to show the effectiveness of the model.

2.3: Game Theory

Game theory attempts to model players' behavior in strategic situations where a player makes decisions based upon other players' decisions. According to (Rasmusen, 2007) game theory is a modeling tool that is "concerned with the actions of decision makers who are conscious that their actions affect each other." Situations where decisions are made based upon the decisions of others can be considered a game and thus modeled via game theory. For example, the owners of two competing gas stations choosing the price to sell gas at their stations could be modeled via game theory. Each owner's decision of price to sell their gas affects the other owner's decision. Situations where decision makers do not consider the decisions of others when making their decisions, such as a company interviewing candidates for a job position, are not considered a game and are not modeled via game theory. Game theory examines both cooperative and non-cooperative games. Games where there is competition with no explicit agreement between the players to cooperate, for example two competing retailers, are considered non-cooperative games. Games where coalitions are formed are considered cooperative games. Coalitions are groups formed that are composed of a subset of the players in the game that explicitly cooperate in decision making rather than competing as in a non-cooperative game.

(Rasmusen, 2007) describes four essential components of a model based on non-cooperative game theory. The four essential components are players, actions, payoffs, and information. Players are entities that make decisions. Each player's goal is to maximize his or her expected payoff through his or her decisions. An action is a choice that a player can make. A payoff is the utility that a player receives after all decisions have been made and thus the game has ended. A player's information includes the payoffs of the different variables and the knowledge of the actions that have previously been taken. Together the players, actions, payoffs, and information allow for the adequate description of a game in order for it to be modeled.

John F. Nash has made many contributions to the area of game theory through (Nash, 1950), titled "The Bargaining Problem", which examined a two-person bargaining situation, and (Nash, 1951), titled "Non-Cooperative Games", which led to the term "Nash equilibrium." (Nash, 1951) examines a scenario where the players act independently of each other, meaning that no coalitions are formed. The contributions made by (Nash, 1951) include defining equilibrium points and proving that a finite non-cooperative game always has one or more equilibrium points. The equilibrium points described in (Nash, 1951), which are also mentioned in (Nash, 1950b), would later be known as a Nash equilibrium. (Epstein, 2009) defines a Nash equilibrium as "a set of strategies and corresponding payoffs such that no player can benefit by changing his strategy while the strategies of all other players are held invariant." Additionally (Epstein, 2009) explains that at least one mixed strategy Nash equilibrium exists for games with a finite amount of players and a finite set of strategies. Additionally the Nash equilibrium for zero-sum games is a minimax equilibrium.

In non-cooperative game theory, response functions describe the response of one player to the other player's decisions. A response function shows the action a player will take with the knowledge of the other player's action. In a non-cooperative game, the response function describes how a player reacts to the opposing player, since there is no opportunity for cooperation between the players. The response functions are important to the competitive transshipment problem because they indicate what one retailer's actions will be in response to the other's actions. Response functions can be plotted simultaneously in order to determine the actions for all players that constitute a Nash equilibrium. Consider a situation similar to (Rudi, 2001) where two retailers must choose order quantities in a competitive game situation. Plotting the two response functions (response of player 1 to the actions of player 2, and the response of player 2 to the actions of player 1) graphically reveals the set of equilibria for the two retailers for a specific set of parameters. Using the graphs it can be seen that the retailers' choice of ordering quantity will converge to one of the intersections shown in the graphs. Figure 10 through Figure 13 demonstrate the convergence process to the equilibrium point by way of the graphs of the retailers' response functions. Q_1 indicates an action (order quantity) chosen by retailer 1 and Q_2 indicates an action chosen by retailer 2.

As seen in Figure 10, if retailer 1 were to choose a (non-equilibrium) quantity of $Q_1 = 70$, then the figure shows how retailer 2's response would be a (non-equilibrium) quantity of (approximately) 135. Figure 11 subsequently demonstrates how retailer 1 would respond to the retailer 2's choice of 135, with a (non-equilibrium) quantity of (approximately) 109. As this is still not an equilibrium quantity, retailer 2 would respond with a quantity of approximately 120 (as seen in Figure 12). The evolution of this conceptual adjustment process continues in Figure 13, as the choices converge on the equilibrium point A, where the two response functions cross. In this example, because of the symmetry of the response functions, $Q_1 = Q_2$ at the equilibrium point.

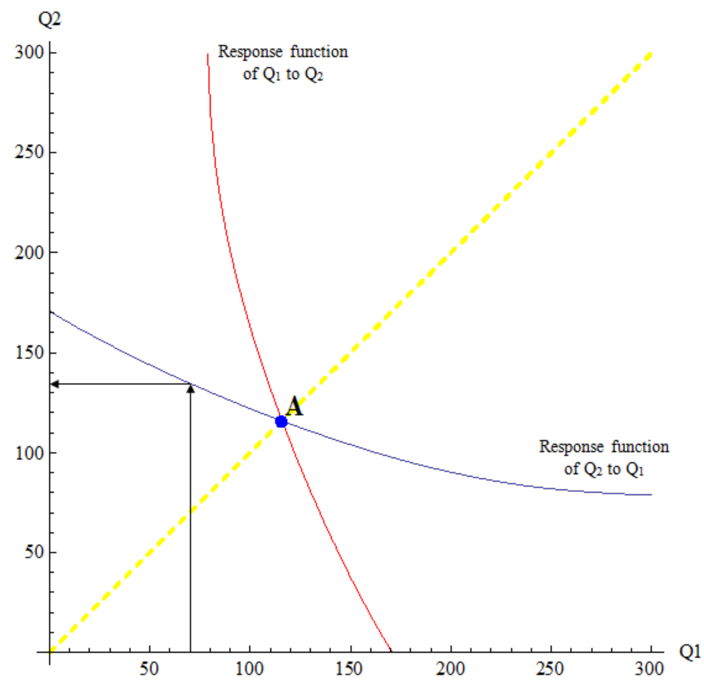

Figure 10: Retailer 2's response to retailer 1's choice of an ordering quantity of 70.

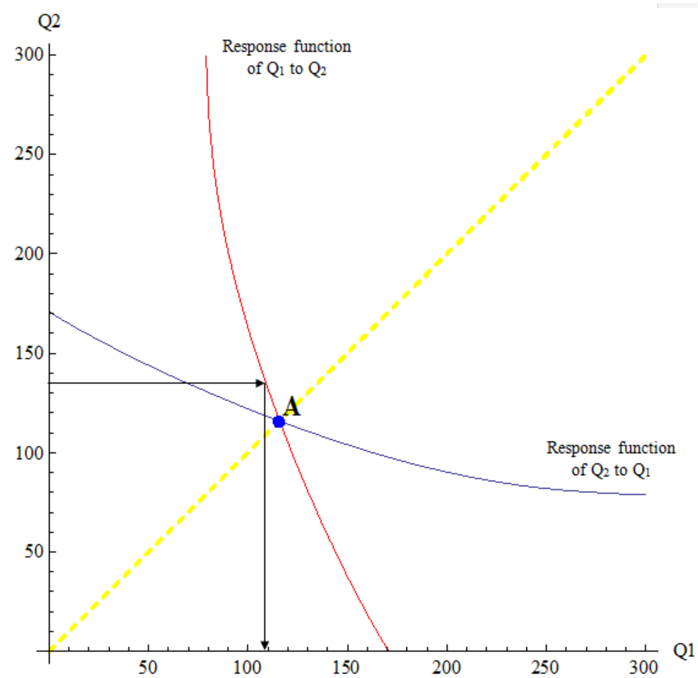

Figure 11: Retailer 1's response to retailer 2's choice of an ordering quantity of 135.

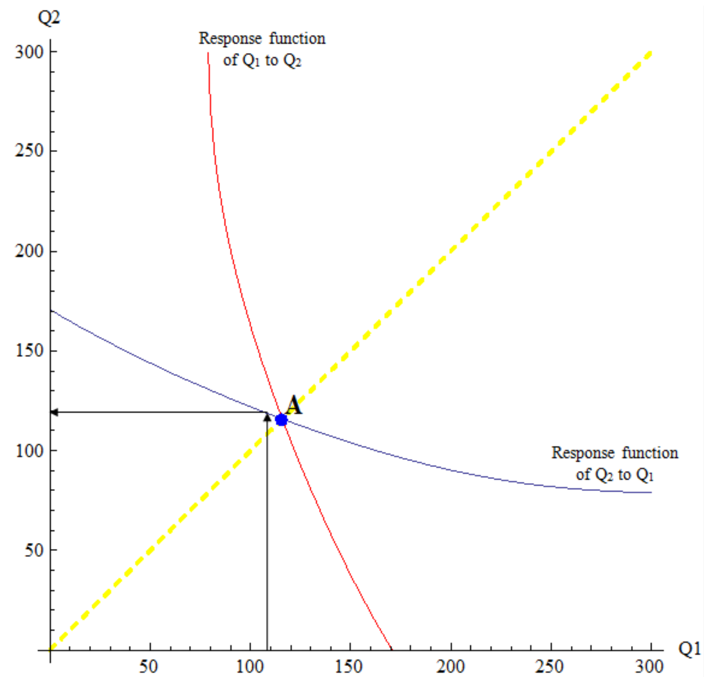

Figure 12: Retailer 2's response to retailer 1's choice of an ordering quantity of 109. The quantities are converging to Intersection A.

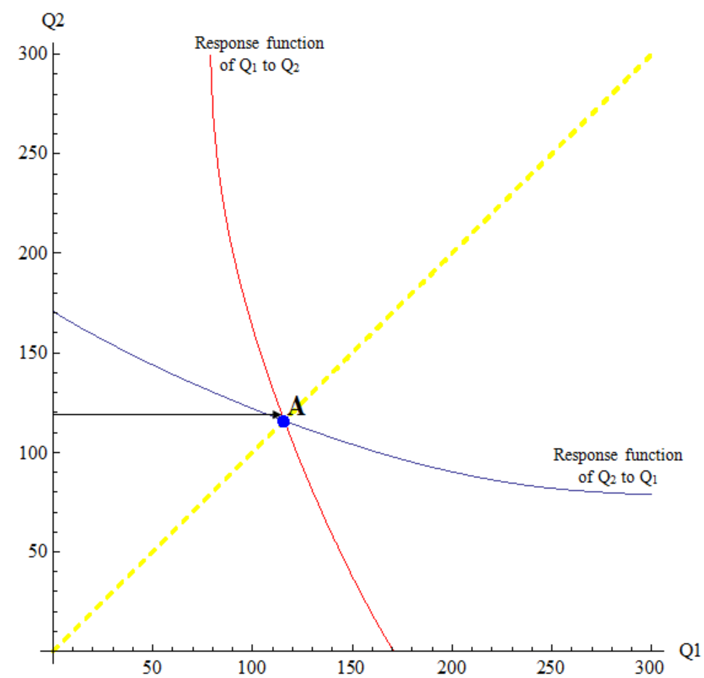

Figure 13: Retailer 1's response to retailer 2's choice of ordering quantity of 120. Nearly converged on Intersection A.

When appropriate cost conditions are in place, there exists a unique Nash equilibrium for the combination of response functions for two retailers for the situation considered in (Rudi, 2001). The conditions for the existence of a unique Nash equilibrium are given by (Fudenberg, 1991) and are also discussed in (Rudi, 2001). In order to establish the existence of a unique Nash equilibrium, it must be shown that the response functions are monotonic and that the absolute value of the slope of the response function is less than one.

In (Sosic, 2006) a myopic and farsighted stability is examined in a transshipment model using the framework of cooperative game theory. This is different than the previous research discussed in this chapter because a cooperative game is being examined rather than a non-cooperative game. A myopic framework assumes that the retailers only look at the immediate consequences to their actions whereas in a farsighted framework the retailers look at the potential subsequent reactions that may occur as a result of their actions and the resulting changes in profit allocations. It is concluded that a grand coalition (grouping that includes all retailers) is the only coalition structure that is stable under all possible relationships between unsold inventories and unsatisfied demands when using a farsighted framework.

In (Anupindi, 2001) a "coopetitive" framework is developed in order to address the sequential inventory and allocation decisions. In the "coopetitive" framework each retailer first determines their inventory and then demand is satisfied. Next the retailers cooperate by sharing the remaining inventories to satisfy unmet demand and allocate the profit generated from the inventory sharing. In (Granot, 2003) a similar framework is used that includes an additional non-cooperative stage where each retailer determines the amount of residuals that they want to share with the other retailers. The work in (Sosic, 2006) is based on the model from (Granot, 2003).

(Slikker 2005) examines a scenario with *n* retailers, where each is facing a newsvendor problem. The research presented is focused on determining if groups of transshipping retailers can improve their expected joint profit by coordinating their orders and whether or not the retailers should participate in transshipping. The situation is analyzed by defining a cooperative game, called a general news-vendor game. It is found that the general news-vendor games have non-empty cores. A non-empty core means that no group of players (retailers) has an incentive to not cooperate.

The model described in this dissertation varies from the cooperative game models in that the retailers are competitively driven when making inventory decisions, taking into consideration the other retailer's inventory decisions. In the cooperative game models an explicit agreement is required to manage the joint profits among the players. In the non-cooperative model studied here, the supplier's quantity discount decisions lead to equal expected profits for the retailers in equilibrium without an explicit profit-sharing agreement. Additionally, this work establishes results for the two-block tariff quantity discount structure. To our knowledge, no other work based on the cooperative game model has existing results for this quantity discount structure.

Chapter 3: Transshipment Game with Two Retailers and Quantity Discounts

This chapter first develops the model for a decentralized two retailer transshipment game based on the classic newsvendor model. Quantity discounts are then introduced to this model for a transshipment game with two retailers. The expected cost function is developed, and the optimal actions of the retailers are described through the development of the response functions, based on derivatives of the costs for the retailers. Sample graphs are given demonstrating the existence of equilibrium points as a function of the discount schedule offered by the supplier to the retailers. Additionally, expressions for the profits from both a retailer's perspective and a supplier's perspective are developed. Studying the potential equilibrium points and how they relate to profit leads to new ways of viewing the influence of the supplier in guiding the behavior of the retailers through the supplier's choice of the discount schedule.

3.1: The Model

The basis for the model is the newsvendor model where inventory is purchased at the beginning of each time period, can be salvaged at the end of each time period, and can no longer be sold during the following time periods. The expected profit in the newsvendor model is defined as the following:

$$\pi^n = E\{r * \min(D, Q) + s * \max(Q - D, 0) - p * \max(D - Q, 0)\} - cQ \tag{1}$$

where π^n is the expected profit for the newsvendor which is represented by the superscript n. The decision-maker in the newsvendor orders with information about the distribution of demand, but without knowing its realized value. The demand that the newsvendor faces in each time period and the quantity that the newsvendor has ordered in each time period are represented by D and Q respectively. The cost per unit of inventory that the newsvendor purchases from the supplier is represented by c, where c is constant cost for all units purchased and $c > 0$. The revenue to the newsvendor per unit sold is represented by r, where $r > c$. If the newsvendor has excess inventory at the end of a time period it can be salvaged for a value of s, where $s < c$. If the newsvendor fails to meet demand (i.e. $D > Q$), a penalty cost p is paid per each unit of demand not met where $p \geq 0$. Taking the derivative of Equation 1 with respect to Q results in the marginal profitability of Q:

$$\frac{\partial \pi^n}{\partial Q} = r\,\Pr(D > Q) + s\,\Pr(D < Q) + p\,\Pr(D > Q) - c \qquad (2)$$

Assuming that the cumulative demand distribution is continuous and strictly increasing, a unique order quantity Q_n exists such that the newsvendor's profit is at a maximum. Setting Equation 2 equal to zero and rearranging results in the following:

$$\Pr(D < Q^*) = \frac{r+p-c}{r+p-s} \qquad (3)$$

where Q^* is the profit-maximizing order quantity.

The retailer model (similar to that in (Rudi, 2001)) is based on the newsvendor, but expands on it by adding a second newsvendor to the model and allowing the two newsvendors to transship inventory to each other. The two newsvendors face separate and independent demands and still can salvage excess inventory and incur penalty costs as in the newsvendor model. The transshipment allows for a newsvendor's unmet demand to be met by the other newsvendor's excess inventory, where the ordering quantity is determined by each newsvendor. The expected profit at location *i* is:

$$\pi_i^d(Q_i, Q_j) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} - c_n Q_i \tag{4}$$

where subscript *j* indicates the competing retailer (also a newsvendor). π_i^d represents the expected profit at location *i* in a decentralized newsvendor transshipment system. The price charged by retailer *i* for each unit transshipped to retailer *j* is represented by c_{ij}, likewise, the price charged by retailer *j* for each unit transshipped to retailer *i* is represented by c_{ji}. The costs associated with shipping each unit from retailer *i* to retailer *j* is represented by τ_{ij}, and is incurred by retailer *i*. The amount of inventory to be transshipped from retailer *i* to retailer *j* is represented as T_{ij}, and is defined as:

$$T_{ij} = \min[\max(D_j - Q_j, 0), \max(Q_i - D_i, 0)] \tag{5}$$

Retail sales for retailer *i* are defined as:

$$R_i = \min(D_i, Q_i) + T_{ji} \tag{6}$$

Unsold stock for retailer *i* is defined as:

$$U_i = \max(Q_i - D_i - T_{ij}, 0) \tag{7}$$

And unmet demand for retailer *i* is defined as:

$$Z_i = \max(D_i - Q_i - T_{ji}, 0) \tag{8}$$

A few assumptions are necessary to ensure that transshipments occur naturally. In order to ensure that it is profitable to send a transshipment, the transshipment price must be greater than the salvage value plus the cost of sending the unit to the other retailer $c_{ij} > s_i + \tau_{ij}$. In order to ensure that it is profitable to receive transshipments, the transshipment price must be less than the marginal value of an additional sale $c_{ij} < r_j + p_j$. Additionally, it is assumed that the two retailers are symmetric in cost structure so that it is not beneficial to always purchase the inventory for both retailers through one retailer and then transship to the other retailer.

The possible transshipment events are defined in Table 1:

Event	Description	Pr{Event}	Transshipments
E_{LowD}^{0}	$D_1 < Q_1, D_2 < Q_2$	$\alpha(Q_1, Q_2)$	$T_{12} = 0, T_{21} = 0$
E_{HighD}^{0}	$Q_1 < D_1, Q_2 < D_2$	$\varphi(Q_1, Q_2)$	$T_{12} = 0, T_{21} = 0$
$E_{Q_1 - D_1}^{12}$	$Q_1 + Q_2 - D_2 < D_1 < Q_1$	$\beta_1(Q_1, Q_2)$	$T_{12} = Q_1 - D_1, T_{21} = 0$
$E_{D_1 - Q_1}^{21}$	$Q_1 < D_1 < Q_1 + Q_2 - D_1$	$\beta_2(Q_1, Q_2)$	$T_{12} = 0, T_{21} = D_1 - Q_1$
$E_{D_2 - Q_2}^{12}$	$Q_2 < D_2 < Q_1 + Q_2 - D_1$	$\gamma_1(Q_1, Q_2)$	$T_{12} = D_2 - Q_2, T_{21} = 0$
$E_{Q_2 - D_2}^{21}$	$Q_1 + Q_2 - D_1 < D_2 < Q_2$	$\gamma_2(Q_1, Q_2)$	$T_{12} = 0, T_{21} = Q_2 - D_2$

Table 1: Events Table

A graphical representation of the transshipment events for the transshipment model is shown in Figure 14.

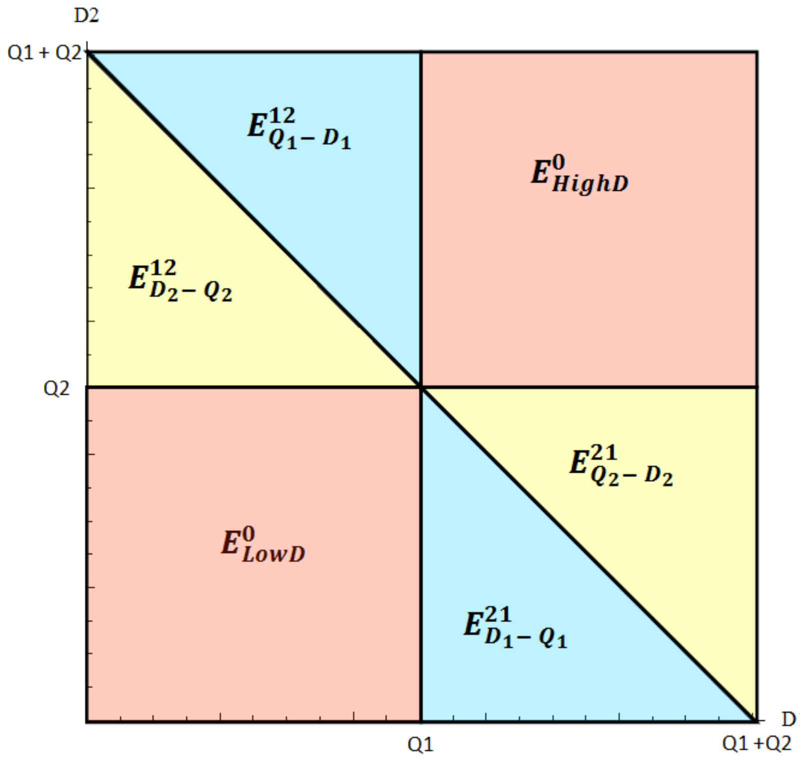

Figure 14: Graphical Illustration of Events in transshipment model

The pure transshipment model is extended here to include a quantity discount based on a two-block tariff from the supplier to the retailer. The quantity above which the discount is offered to the retailer is represented by x. In this discount model, the normal price charged by the supplier to the retailers per unit is represented by c_n and the reduced price charged by the supplier to the retailers per unit over x is represented by c_r, where $c_r < c_n$. The expected profit for retailer i for the quantity discount newsvendor transshipment model is represented by π_i^{qd} and is defined as:

$$\pi_i^{qd} = \begin{cases} \pi_i^d, & Q_i \le x \\ \pi_i^r, & Q_i > x \end{cases} \tag{9}$$

where π_i^d is:

$$\pi_i^d(Q_i, Q_j) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} - c_i Q_i$$

and π_i^r is the expected profit where retailer i is ordering enough units to receive some units at a discounted price from the supplier and is defined as:

$$\pi_i^r(Q_i, Q_j) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} - x(c_n - c_r) - c_r Q_i. \tag{10}$$

In addition to the other assumptions assume that $c_r > s$. This assumption ensures that it is not profitable for a retailer to purchase stock and salvage it for a profit. Using the definitions in Equations 5, 6, 7, and 8 in Equation 10 results in the following:

$$\begin{aligned}\pi_i^r(Q_i, Q_j) = E\{&r_i[\min(D_i, Q_i) + \min[\max(D_i - Q_i, 0), \max(Q_j - D_j, 0)]]\\ &- c_{ji}[\min[\max(D_i - Q_i, 0), \max(Q_j - D_j, 0)]]\\ &+ (c_{ij} - \tau_{ij})[\min[\max(D_j - Q_j, 0), \max(Q_i - D_i, 0)]] + s_i[\max(Q_i\\ &- D_i - \min[\max(D_j - Q_j, 0), \max(Q_i - D_i, 0)])] - p_i[\max(D_i - Q_i\\ &- \min[\max(D_i - Q_i, 0), \max(Q_j - D_j, 0)])] - x(c_n - c_r) - c_r Q_i\end{aligned} \tag{11}$$

Figure 15 shows the transshipment events that occur within the transshipment with quantity discount model. Note that we explicitly allow that the quantity discounts could lead to non symmetric order quantities, and this significantly affects the defined event regions as compared to the graphical illustration of the model without quantity discounts.

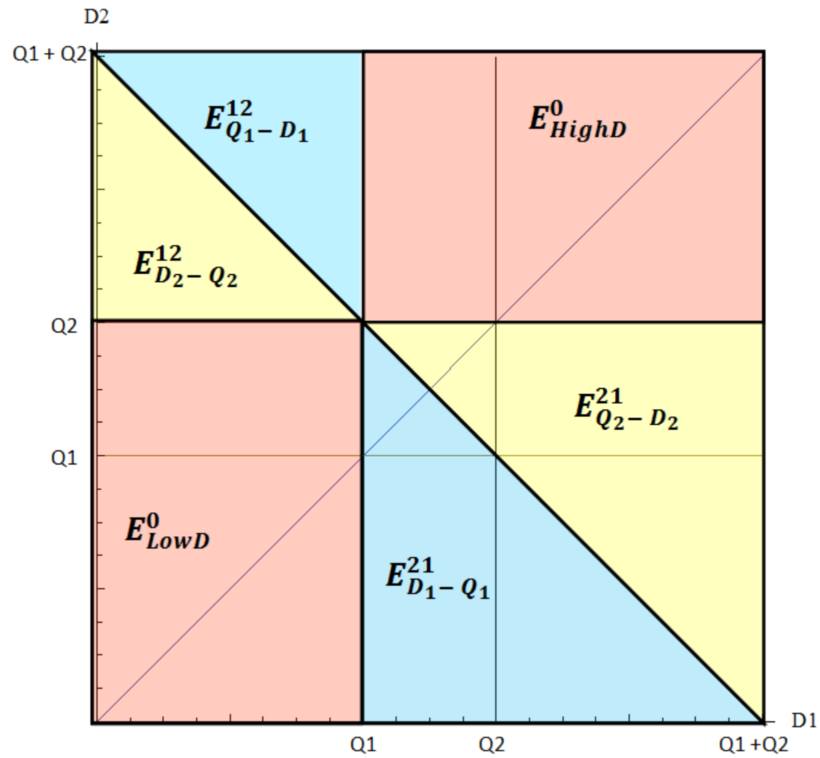

Figure 15: Graphical illustration of the transshipment events that occur in the transshipment with quantity discount model.

Rearranging the terms from Equation 11 and expanding out the expectation operator results in the following:

$$
\begin{aligned}
\pi_i^r(Q_i, Q_j) = r_i \Bigg[& \int_0^{\infty} \int_0^{Q_i} D_i f(D_i) f(D_j) dD_i dD_j + \int_0^{\infty} \int_{Q_i}^{\infty} Q_i f(D_i) f(D_j) dD_i dD_j \\
& + \int_0^{Q_j} \int_{Q_i}^{Q_i+Q_j-D_j} (D_i - Q_i) f(D_i) f(D_j) dD_i dD_j \\
& + \int_0^{Q_j} \int_{Q_i+Q_j-D_j}^{\infty} (Q_j - D_j) f(D_i) f(D_j) dD_i dD_j \Bigg] \\
& - c_{ji} \Bigg[\int_0^{Q_j} \int_{Q_i}^{Q_i+Q_j-D_j} (D_i - Q_i) f(D_i) f(D_j) dD_i dD_j \\
& + \int_0^{Q_j} \int_{Q_i+Q_j-D_j}^{\infty} (Q_j - D_j) f(D_i) f(D_j) dD_i dD_j \Bigg] \\
& + (c_{ij} - \tau_{ij}) \Bigg[\int_0^{Q_i} \int_{Q_i+Q_j-D_i}^{\infty} (Q_i - D_i) f(D_i) f(D_j) dD_i dD_j \\
& + \int_0^{Q_i} \int_{Q_j}^{Q_i+Q_j-D_j} (D_j - Q_j) f(D_i) f(D_j) dD_i dD_j \Bigg] \\
& + s_i \Bigg[\int_{Q_j}^{Q_i+Q_j} \int_0^{Q_i+Q_j-D_j} (Q_i - D_i + Q_j - D_j) f(D_i) f(D_j) dD_i dD_j \\
& + \int_0^{Q_j} \int_0^{Q_i} (Q_i - D_i) f(D_i) f(D_j) dD_i dD_j \Bigg] \\
& - p_i \Bigg[\int_0^{Q_j} \int_{Q_i+Q_j-D_j}^{\infty} (D_i + D_j - Q_i - Q_j) f(D_i) f(D_j) dD_i dD_j \\
& + \int_{Q_j}^{\infty} \int_{Q_i}^{\infty} (D_i - Q_i) f(D_i) f(D_j) dD_i dD_j \Bigg] - x(c_n - c_r) - c_r Q_i
\end{aligned}
\tag{12}
$$

Next we define:

$$g_i(Q_i, Q_j) = r_i\left[\int_0^{\infty}\int_{Q_i}^{\infty} f(D_i)f(D_j)dD_idD_j - \int_0^{Q_j}\int_{Q_i}^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_idD_j\right]$$
$$+ c_{ji}\left[\int_0^{Q_j}\int_{Q_i}^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_idD_j\right]$$
$$+ (c_{ij} - \tau_{ij})\left[\int_0^{Q_i}\int_{Q_i+Q_j-D_i}^{\infty} f(D_i)f(D_j)dD_idD_j\right]$$
$$+ s_i\left[\int_{Q_j}^{Q_i+Q_j}\int_0^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_idD_j\right.$$
$$\left.+ \int_0^{Q_j}\int_0^{Q_i} f(D_i)f(D_j)dD_idD_j\right]$$
$$+ p_i\left[\int_0^{Q_j}\int_{Q_i+Q_j-D_j}^{\infty} f(D_i)f(D_j)dD_idD_j - \int_{Q_j}^{\infty}\int_{Q_i}^{\infty} f(D_i)f(D_j)dD_idD_j\right] \tag{13}$$

Differentiation of Equation 12 with respect to Q_i results in the following equation:

$$\frac{\partial \pi_i^r}{\partial Q_i} = g_i(Q_i, Q_j) - c_r \tag{14}$$

A similar result is derived from π_i^d when differentiation with respect to Q_i is applied to Equation 4, as shown in the following equation:

$$\frac{\partial \pi_i^d}{\partial Q_i} = g_i(Q_i, Q_j) - c_n \tag{15}$$

Equation 15 has the same structural form as Equation 9 from (Rudi, 2001). (Rudi, 2001) studies this equation to show that the response functions are monotonic and that the absolute value of the slope is less than one, thus proving that there exists a unique Nash equilibrium for any pair of competing response curves.

Setting the derivatives in Equation 14 and Equation 15 equal to zero and solving for Q_1 as a function of Q_2 and Q_2 as a function of Q_1 results in the optimal ordering quantities for each retailer (Q_1^*, Q_2^*) as a function of the other retailer's ordering quantity. Thus Equation 16 and Equation 17 define the response functions for retailer *i*.

$$g_i(Q_i^*, Q_j^*) = c_r \tag{16}$$

$$g_i(Q_i^*, Q_j^*) = c_n \tag{17}$$

The optimal ordering quantity points that are common to both retailers' response functions define the equilibrium points for the two retailers' ordering quantities. A major difference between the model presented in this dissertation compared to the model in (Rudi, 2001) is the possible existence of up to four potential equilibrium points depending on the value of the quantity discount, x. These four potential equilibrium points are a result of the "family" of two response functions, given by Equation 16 and Equation 17, for each retailer. The combinations of the response functions, one for each retailer, result in four possible intersections which are labeled A(Q_{1A}^*, Q_{2A}^*), B(Q_{1B}^*, Q_{2B}^*), C1(Q_{1C1}^*, Q_{2C1}^*), and C2(Q_{1C2}^*, Q_{2C2}^*).

3.1.1: The Exponential Demand Distribution Case

A special case of the two retailer transshipment with quantity discounts problem is the case where the exponential demand distribution is used. Unlike the other demand distributions used in this dissertation, the characteristics of the exponential distribution allow the expected retailer profit function, and subsequently the retailer response function to be simplified. Equation 18 gives the retailer profit when buying according to the reduced cost from the supplier and an exponential demand distribution is used, $\pi_i^{r,expo}$. Note that a similar equation can derived when using the normal cost from the supplier, where the only difference is the cost from the supplier. The variable *m* is defined as one divided by the mean of the exponential distribution.

$$\pi_i^{r,expo} = \frac{e^{-m(Q_i+Q_j)}}{m}\Big[-p\big(1+mQ_j\big) + c\left(e^{mQ_i} - e^{mQ_j} + m\big(Q_j - Q_i\big)\right)$$
$$+ r\Big(e^{m(Q_i+Q_j)} - mQ_j - 1\Big)$$
$$+ s\Big(1 - e^{mQ_i} + e^{mQ_j} - e^{m(Q_i+Q_j)} + mQ_i + e^{m(Q_i+Q_j)}mQ_i\Big)$$
$$+ \ \tau(1 - e^{mQ_i} + mQ_i) + c_r e^{m(Q_i+Q_j)}m(x - Q_i) - c_n e^{m(Q_i+Q_j)}mx\Big] \tag{18}$$

Compared to Equation 12, it is easy to see that this expression is much smaller in size and the integration has been simplified. Equation 19 gives the response function for the retailer when using the reduced cost from the supplier.

$$e^{-m(Q_i+Q_j)}\Big[(r+p)\big(1+mQ_j\big) + c\left(-1 + e^{mQ_j} + m\big(Q_i - Q_j\big)\right)$$
$$+ s\Big(-e^{mQ_j} + e^{m(Q_i+Q_j)} - mQ_i\Big) - mQ_i\tau\Big] = c_r \tag{19}$$

Like the expected profit equation, the expression for the response function for the exponential case is much smaller in size then its general counterpart presented in Equation 16.

3.2: Supplier Profit

The expected supplier profit π^s, is given by Equation 20, where c_s is the supplier's production cost.

$$\pi^s = \big(Q_i + Q_j\big) * c_r + (c_n - c_r) * \big[\mathrm{Max}(Q_i, x) + \mathrm{Max}\big(Q_j, x\big)\big] - \big(Q_i + Q_j\big) * c_s \tag{20}$$

3.3: Results

3.3.1: Analysis of Potential Equilibrium

By differentiating the profit model without a quantity discount for a single retailer with respect to that retailer's ordering quantity Q_1, we get an equation that describes the order quantity of the first retailer Q_1 as a function of the ordering quantity of the second retailer Q_2. Note that the notation is switching from the more generalized notation (*i* and *j*) to a more specific notation (1 and 2). Using Equation 12 to graph Q_1 as a function of Q_2 and also graphing Q_2 as a function of Q_1, we obtain the graph shown in Figure 16. The parameters for this figure are based on: $c_{ij} = c_{ji} = 26$; $c_n = 20$; $c_r = 13$; $r_i = 40$; $s_i = 10$; $p_i = 0$; $\tau_{ij} = 2$; D_i , D_j ~ Normal (mean =100, st. dev. = 50). Additionally, *x* is very large in this figure in order to prevent the retailers from receiving a quantity discount. Figure 16 is similar to Figure 2 from (Rudi, 2001).

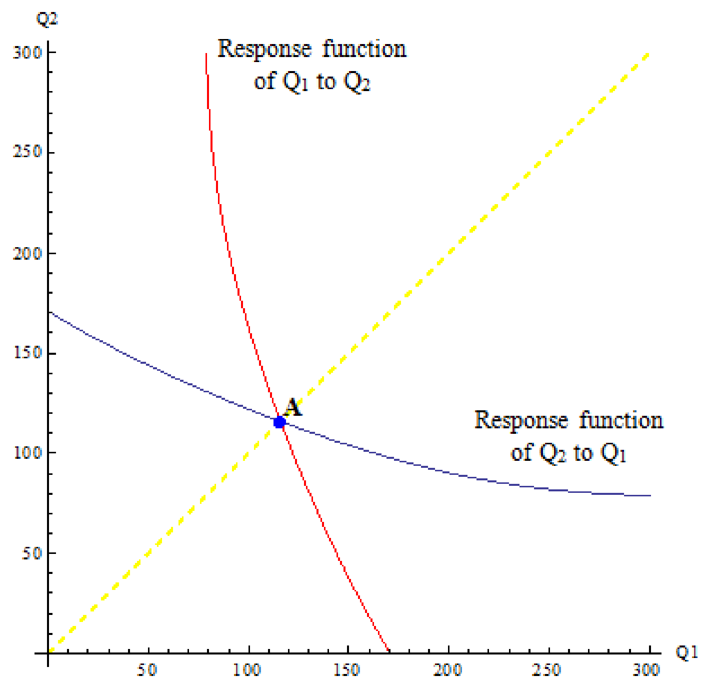

Figure 16: Equilibrium point A for a two retailer transshipment model without quantity discount.

Note that retailer 1 and retailer 2 are symmetric in their associated costs in this example. Because of this the graph of the response function for each retailer is a reflection of the other retailer's response function along the line $Q_1 = Q_2$. The intersection made by the two lines at point A represents a potential equilibrium. In a Nash equilibrium each retailer will keep their order quantities unchanged given the knowledge of the other retailers order quantity. Additionally, note that in the model in (Rudi 2001) when the two retailers are symmetric in their costs, the point at which the Nash equilibrium occurs will be a symmetric point where $Q_1 = Q_2$.

Unlike in Figure 16, when the quantity discount is taken into account, each retailer has an overall response function that comes about through the composition of two curves based on Equation 16 and Equation 17. The two curves do not intersect each other and are formed individually, one for each cost per unit (c_n and c_r, below and above the discount quantity, respectively). Figure 17 shows the two curves for one of the retailers. When a quantity discount exists the overall response function has a discontinuity at the discount triggering quantity of the quantity discount. Figure 18 shows the overall response function for a single retailer with the discontinuity created by the quantity discount.

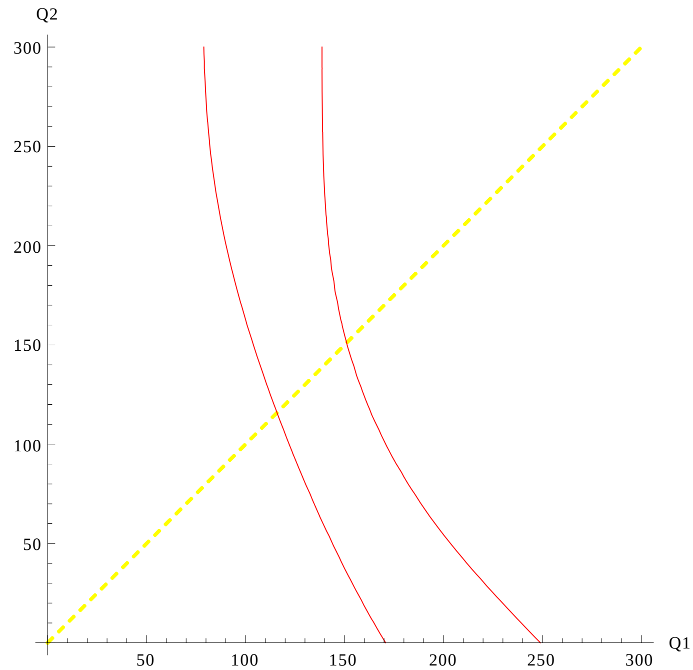

Figure 17: Two curves that determine the response function for a single retailer.

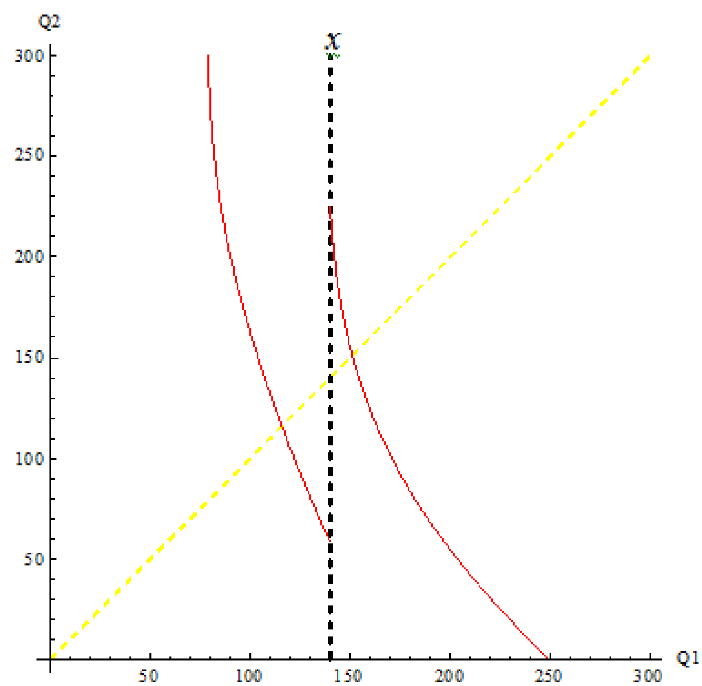

Figure 18: Overall response functions for a single retailer with discontinuity from quantity discount at a quantity of *x* = 140

Plotting both sets of response functions simultaneously (response of Q_1 to Q_2 and the response of Q_2 to Q_1) in a model that allows for a quantity discount from the supplier is shown in Figure 19.

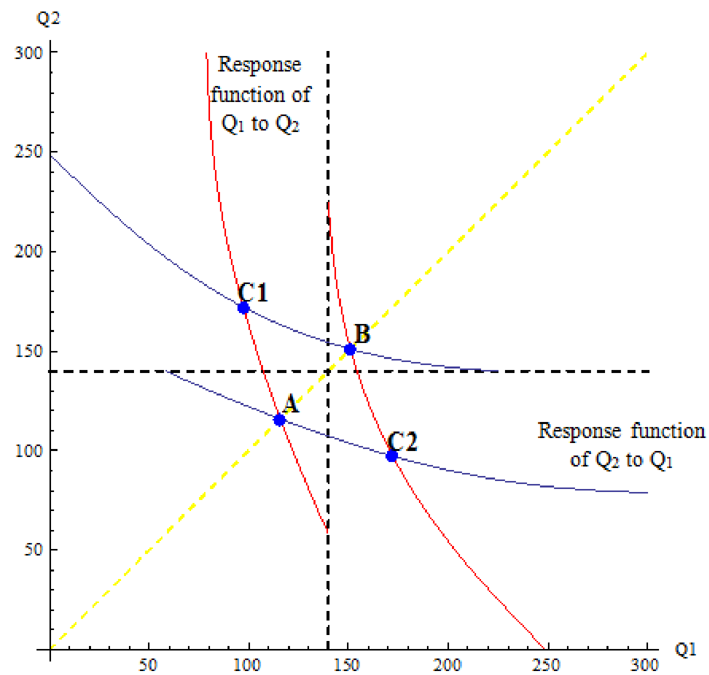

Figure 19: Plot of intersecting response functions showing the existence of four equilibrium points for two retailers with a quantity discount at *x* = 140.

Notice that there are now four potential Nash equilibrium points where as without the quantity discount there exists just a single Nash equilibrium. Two of the four intersections, Intersection A and Intersection B are symmetric ($Q^*_{1A} = Q^*_{2A}$ and $Q^*_{1B} = Q^*_{2B}$) and occur on the line $Q_1 = Q_2$. The other two intersections, Intersection C1 and Intersection C2 are not symmetric and do not lie on the line $Q_1 = Q_2$. The value of Q_1 in Intersection C1 is equal to the value of Q_2 in Intersection C2 and vice versa, that is $Q^*_{1C1} = Q^*_{2C2}$ and $Q^*_{1C2} = Q^*_{2C1}$. The existence of the four intersections depends on the value of the discount triggering quantity *x*. Figure 20 shows a graph of the response functions of a transshipment model with quantity discount where *x* is a large value. Thus with large values of *x*, only Intersection A exists; this is because the discount triggering quantity is too large for the retailers to take advantage of. When *x* decreases to the point at which *x* is equal to or less than the value of Q_2 in Intersection C1 and Q_1 in Intersection C2, as shown in Figure 21, the existence of Intersection C1 and Intersection C2 are present.

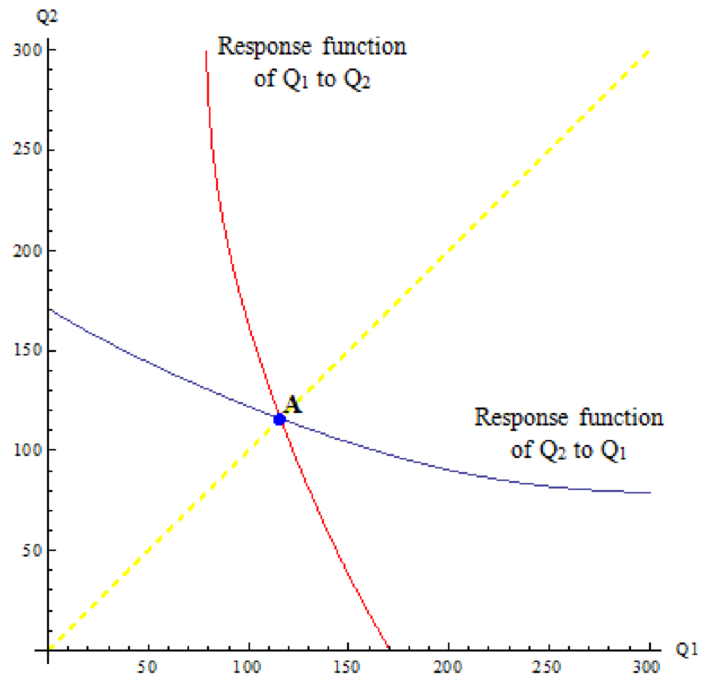

Figure 20: Transshipment model with quantity discount where *x* is large.

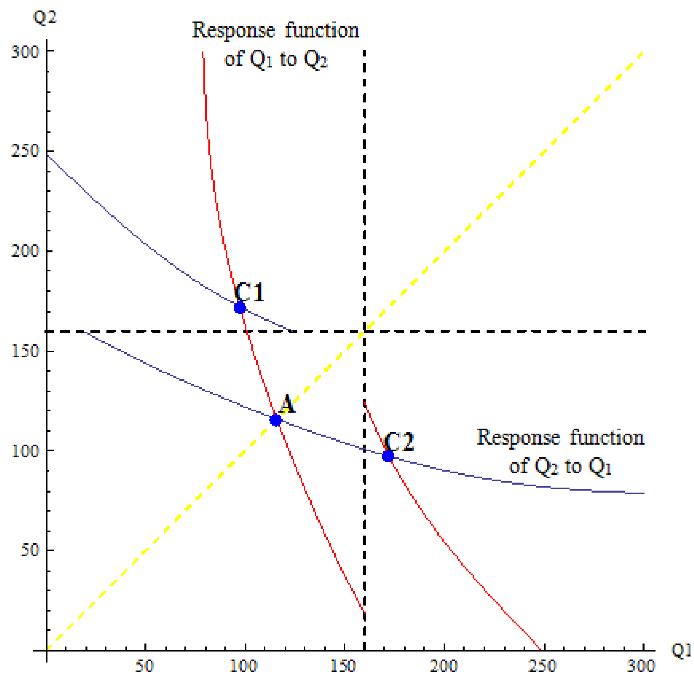

Figure 21: Transshipment model with quantity discount, *x* = 160. Intersections A, C1, and C2 present.

As x continues to decrease, the existence of Intersection B as well as the other three intersections are present (Figure 19). Intersection B exists at the point at which x is equal to the value of Q_1 and Q_2 from Intersection B. Once the value of x has decreased below the value of Q_1 and Q_2 from Intersection A, Intersection A no longer exists as shown in Figure 22.

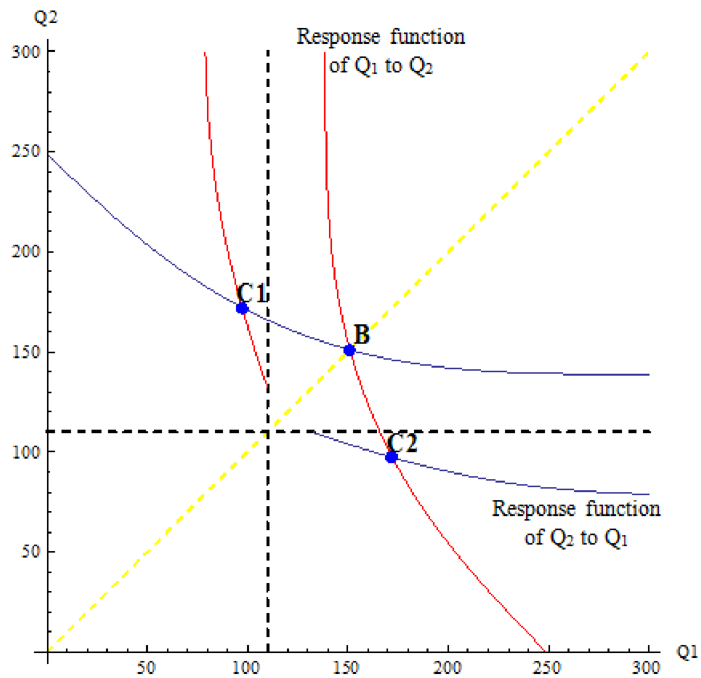

Figure 22: Transshipment model with quantity discount, *x* = 110.

The last change in existence of the intersections occurs when the value of *x* decreases to a value below the value of Q_1 in Intersection C1 and Q_2 in Intersection C2, at which the intersections C1 and C2 no longer exist as shown in Figure 23. The only existing intersection is now Intersection B.

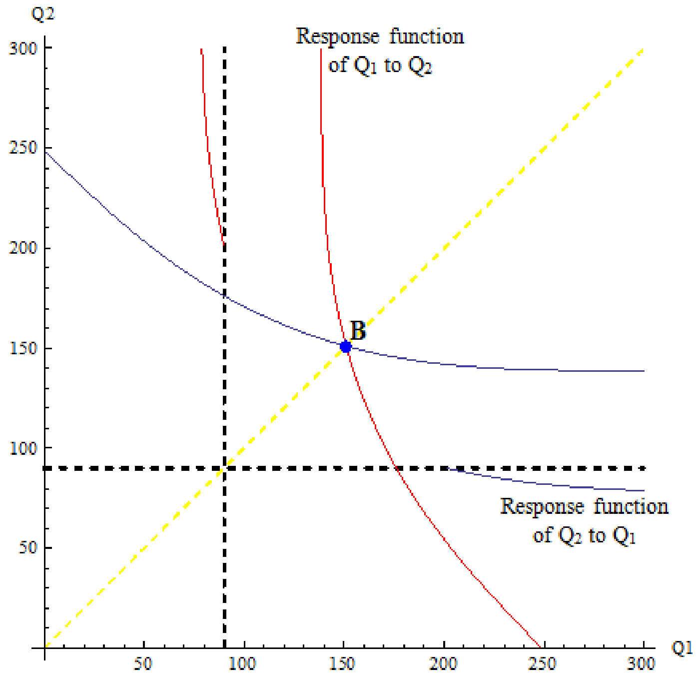

Figure 23: Transshipment model with quantity discount, *x* = 90.

The existence of the four intersections is summarized in Table 2

Name	Symmetric/Non-Symmetric Q's	Range of Existence
A	Symmetric	$Q_{1A}^* = Q_{2A}^* \le x$
B	Symmetric	$x \le Q_{1B}^* = Q_{2B}^*$
C1	Non-Symmetric	$Q_{1C1}^* = Q_{2C2}^* \le x \le Q_{1C2}^* = Q_{2C1}^*$
C2	Non-Symmetric	$Q_{1C1}^* = Q_{2C2}^* \le x \le Q_{1C2}^* = Q_{2C1}^*$

Table 2: Existence of the Intersections as a function of *x*

The four intersections give the four ordering quantity combinations for Q_1 and Q_2 that result in a potential Nash equilibrium.

Proposition 1: A unique intersection exists for any pairing of response functions in Equation 16 and Equation 17, one for each retailer.

Brief Proof: For the symmetric cases the existence of a unique intersection for each pairing of the response functions in (16) and (17) (one for each retailer) is guaranteed when the response functions are monotonic and that the absolute value of the slope is less than 1 ((Fudenberg, 1991), (Rudi, 2001). Through implicit differentiation of the response functions in Equations (16) or (17) (Similar to (Rudi, 2001)) the slope of these response function is known to be nonpositive with an absolute value less than 1. The constants in the right hand sides in (16) and (17) (c_n and c_r) differentiate these response functions from each other. These constants do not play a role in the slope of the response functions through implicit differentiation of the response functions, $\frac{\partial \pi_i^r}{\partial Q_i}$ and $\frac{\partial \pi_i^d}{\partial Q_i}$ with respect to Q_j. For the non-symmetric cases (intersection of Equation (16) for one retailer with Equation (17) for the other retailer) the different right hand side constants do not change the property that the response functions must intersect. ■

3.3.2: Analysis of Retailer Profit

Figure 24 shows the optimal profit for the retailers with differing values of x, which is given by Equation 12. The different lines represent the different ordering quantity combinations that result in a potential Nash equilibrium. These possible ordering quantity combinations are defined by the intersections shown in Figure 19. The lines forming the intersections are a result of the following parameters: $c_{ij} = c_{ji} = 26$, $c_n = 20$, $c_r = 13$, $r_i = 40$, $s_i = 10$, $p_i = 0$, $\tau_{ij} = 2$, D_i, D_j ~ Normal (mean = 100, st. dev. = 50). The Normal distribution was truncated so that negative values were eliminated and the remaining probability distributed proportionately over the positive demand values.

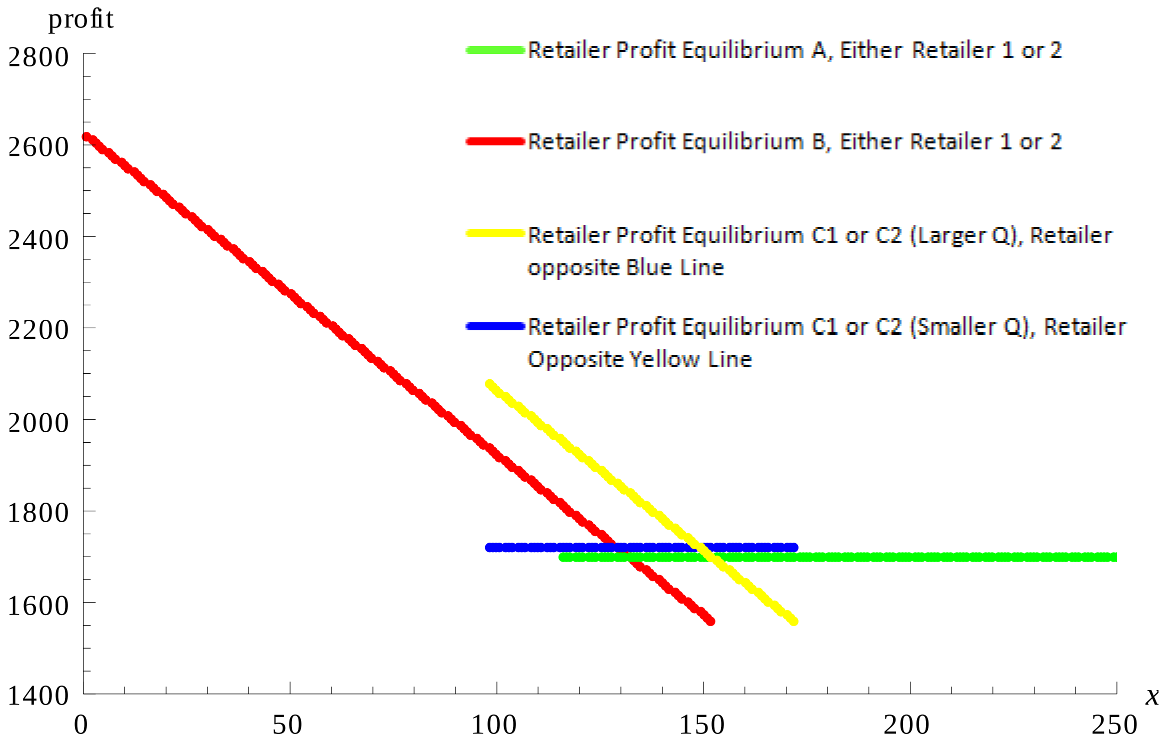

Figure 24: Retailer Profit for different values of *x*, comparing different ordering quantity combinations.

Note that the slope of the lines representing the retailer profit at Equilibrium A and the smaller ordering quantity at either Equilibrium C1 or Equilibrium C2 in Figure 24 will always be zero due to the fact that the ordering quantities for the retailers while at these equilibria is always less than the value of x. Similarly, the slope of the lines representing the retailer profit at Equilibrium B and the larger ordering quantity at either Equilibrium C1 or Equilibrium C2 in Figure 24 will always be equal to $c_r - c_n$ due to the fact that the ordering quantities for the retailers at these equilibria is greater than the value of x and thus the profit is affected as x changes, which can be seen from Equation 12.

In order to determine the point at which the line representing the smaller Q at Equilibrium C1 or C2 and the line representing the larger Q at Equilibrium C1 or C2 cross, a version of Equation 12 for each retailer i is set equal to a version of Equation 12 for retailer j. Rearranging the equation results in Equation 21, which describes the value of x where the profits for both retailers at Equilibrium C1 or C2 are equal.

$$\begin{aligned}
x^{eq} = \Bigg[&(Q_i - Q_j) \\
&* \left[(c - r - p)\int_0^{Q_i}\int_{Q_i+Q_j-D_i}^{\infty} f(D_i)f(D_j)dD_i dD_j + (c - \tau - s)\int_0^{Q_i}\int_{Q_j}^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j\right] \\
&+ \left(-\tau * Q_i + (c - s) * (Q_i + Q_j)\right)\int_{Q_i}^{Q_j}\int_0^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j \\
&+ Q_j(\tau - 2c + 2r + 2p)\int_{Q_i}^{Q_j}\int_{Q_i+Q_j-D_i}^{\infty} f(D_i)f(D_j)dD_i dD_j \\
&+ (2c - 2r - p - \tau)\int_{Q_i}^{Q_j}\int_{Q_i+Q_j-D_i}^{\infty} D_j f(D_i)f(D_j)dD_i dD_j \\
&+ (-2c + \tau + s)\int_{Q_i}^{Q_j}\int_0^{Q_i+Q_j-D_j} D_j f(D_i)f(D_j)dD_i dD_j \\
&+ p\int_{Q_j}^{\infty}\int_{Q_i}^{\infty}(D_j - D_i)f(D_i)f(D_j)dD_i dD_j \\
&+ (-p)\int_{Q_i}^{Q_j}\int_{Q_i+Q_j-D_i}^{\infty} D_i f(D_i)f(D_j)dD_i dD_j \\
&+ s\int_0^{Q_j}\int_0^{Q_i}(D_j - D_i)f(D_i)f(D_j)dD_i dD_j \\
&+ s\int_{Q_i}^{Q_j}\int_0^{Q_i+Q_j-D_j} D_i f(D_i)f(D_j)dD_i dD_j + Q_j(c_r - c_n)\Bigg] / (c_n - c_r)
\end{aligned} \tag{21}$$

Evaluating Equation 21 using the same values for the variables as used in the example above results in $x \approx 148$, which represents the value of x at which the profits for both retailers at Intersection C1 or C2 are equal.

In general, we can state Proposition 1, clarifying the existence of the non-symmetric C equilibrium, followed by a brief conceptual proof:

Proposition 2: If $c_n - c_r$ is large enough, then there exists an x^{eq} that leads to a C equilibrium.

Proof: At either C1 or C2, the profit for the retailer with the larger ordering quantity (the retailer that optimally orders above the quantity discount) is linearly decreasing in *x*, with slope -($c_n - c_r$). Note that the ordering quantities at C1 and C2 do not vary with *x*. The profit for the retailer with the lower profit at C1 or C2 is constant in *x* (because that retailer's order quantity is below x). If $c_n - c_r$ is large enough, then there is scope for enough variation in *x* (if the range of *x* where C1 and C2 exist is wide enough), for the profit of the retailer with the larger order to drop to below the profit for the retailer with the smaller order. The range of existence for C1 and C2 is a function of the distance between the response functions, which is a direct function of $c_n - c_r$. ■

3.3.3: Analysis of Supplier Profit

Supplier profit is given by Equation 20 and depends on x as well as the ordering quantities of the retailers. Figure 25 shows the supplier's profit dependent upon x.

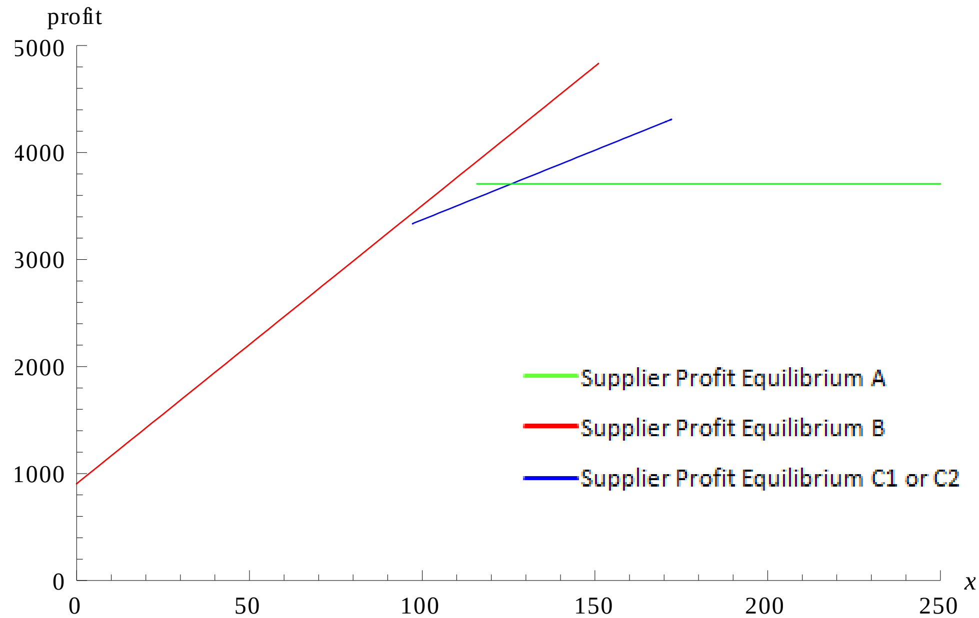

Figure 25: Supplier Profit for various values of *x*.

Note that it can be seen from Equation 12 that the cost is proportional to the value of x, thus the slope of the supplier profit in the low range for x (Equilibrium B) in Figure 25 will always be equal to $2(c_n - c_r)$ due to the fact that the ordering quantities for both of the retailers at the equilibrium is greater than the value of x and thus the supplier profit is affected accordingly. The slope of the supplier profit in the middle range of x (Equilibrium C1 or C2) will always be $c_n - c_r$ due to the ordering quantity for one of the retailers at the equilibria is greater than the value of x while the ordering quantity for the other retailer is below x. The slope of the supplier profit in the high range of x (Equilibrium A) will always be zero due to the ordering quantities for both retailers being less than the value of x and thus there is no change in profit for the supplier for changes in x while at this equilibrium.

Comparing Figure 24 to Figure 25 exposes some important aspects of the model. A point of interest on the graphs is the far left, $x = 0$, which appears to be the best point for the retailers while being the worst point for the supplier. Similarly, the right end of the Equilibrium B supplier profit line appears to be the best point for the supplier while being the worst point for the retailers.

3.3.4: Determining Stability

Determining the recommended ordering quantities requires defining "stable" points or regions for the quantity discount value within the transshipment and quantity discount model. A stable region is defined as a region, distinguished by the discount triggering quantity, where

1. the expected profit for retailer i is equal to the expected profit for retailer j and
2. the two retailers are at an equilibrium that has a larger expected profit for both retailers than any other equilibrium that is in existence for a particular value of x.

We define stable in this way because

1. when the two retailers have equal profit they will not "jockey" for the role in the non-symmetric equilibrium with higher profit.
2. when the two retailers have no other potential equilibrium to switch to that has a higher profit, they will choose the stable equilibrium point.

Using this definition, we can determine the stable regions for the system as shown in Table 3.

Retailer's Choice		
X	**Choice**	**Supplier's Profit Trend**
0 to Q^*_{1C1}	B	Increasing as *x* increases
Q^*_{1C1} to x^{eq}	Unstable	Unstable
x^{eq}	C	-
x^{eq} to Q^*_{2C1}	Unstable	Unstable
Q^*_{2C1} to ∞	A	Constant as *x* increases

Table 3: Retailer's choice defined by stable and unstable regions of *x*

Using the values from our example results in Table 4.

Retailer's Choice		
X	**Choice**	**Supplier's Profit Trend**
0 to 101.26	B	Increasing as *x* increases
101.26 to 147.72	Unstable	Unstable
147.72	C	-
147.72 to 176.75	Unstable	Unstable
176.75 to ∞	A	Constant as *x* increases

Table 4: Stable and unstable regions of *x*

Additionally, Table 4 shows the choice that the retailers will make within each region and the supplier's profit trend within those regions. Here, a retailer will choose the equilibrium point that has the highest expected profit for the retailer. Knowing the retailer's choices within each region allows the supplier to set the best values for the quantity discount. An analysis of the stable retailer's choice for ordering quantities as relates to the supplier is shown in Table 5.

x	Total Quantity Sold	Supplier Profit	Retailer 1 Profit	Retailer 2 Profit
101.261	311.6	2352.7	1874.5	1874.5
147.719	278	2576.9	1702.6	1702.6
176.75-Infinity	238.7	2387.1	1675.1	1675.1

Table 5: Supplier's choice for the value of the quantity discount

As you can see from Table 5 the supplier's profit is highest at the point described by Equation 21, which is the only stable point that occurs in the region where equilibria C1 and C2 exist. This stable point occurs when the retailers are at either equilibrium C1 or C2. The existence of this stable point is dependent upon one of the following two conditions:

$$0 \leq \left(\pi_{cHigh}^{xLow} - \pi_{cLow}^{xLow}\right) \leq (c_{1a} - c_{1b})\left(x_{high} - x_{low}\right) \tag{22}$$

$$0 \leq \left(\pi_{cLow}^{xHigh} - \pi_{cHigh}^{xHigh}\right) \leq (c_{1a} - c_{1b})\left(x_{high} - x_{low}\right) \tag{23}$$

If either Equation 22 or 23 is true then the non-symmetric stable point exists.

3.3.5: Comparison of Models

Table 6 shows example results for the ordering quantities and profits for the three different models: newsvendor, transshipment, and transshipment with quantity discount.

		Ordering Quantity			**Profit**			
Transshipment	**Quantity Discount**	**Retailer 1**	**Retailer 2**	**Total**	**Supplier**	**Retailer 1**	**Retailer 2**	**Total**
No	No	122.6	122.6	245.2	2451.7	1529.9	1529.9	5511.5
Yes	No	119.4	119.4	238.7	2387.1	1675.1	1675.1	5737.3
Yes	Yes	101.3	176.7	278	2576.9	1702.6	1702.6	5982.1

Table 6: Example comparison of different models

Similarly to Table 6, Table 7 shows the percentage of change in results from the newsvendor model.

		Ordering Quantity			Profit			
Transshipment	Quantity Discount	Retailer 1	Retailer 2	Total	Supplier	Retailer 1	Retailer 2	Total
No	No	-	-	-	-	-	-	-
Yes	No	-2.6%	-2.6%	-2.7%	-2.6%	9.5%	9.5%	4.1%
Yes	Yes	-17.4%	44.1%	13.4%	5.1%	11.3%	11.3%	8.5%

Table 7: Percentage of change in results from the newsvendor model

From Table 6 and Table 7 we can draw the conclusion that the transshipment with quantity discounts model is more profitable for both the supplier and the two retailers in comparison to the transshipment model and the newsvendor model. The transshipment model increases retailer profit at the expense of decreasing the supplier profit. The transshipment model with quantity discounts results in increased supplier and retailer profits when compared to either the transshipment model or the newsvendor model. The total ordering quantity of the transshipment model is decreased when compared to the newsvendor model; however, the total ordering in the transshipment with quantity discount model is increased when compared to the other two models.

3.3.6: Parametric Analysis

Some results from the transshipment with quantity discount model with varying input parameters are given in Table 9. Table 8 shows the input parameters for the results shown in Table 9. Note that some of the cases do not result in a stable C equilibrium because they do not meet the conditions described in Equation 22 and Equation 23.

	Variables								
Case	**Distribution**	r	c_{ij}	c_n	c_r	s	τ	p	c_s
1	Normal(200, 50)	40	26	20	13	10	2	5	10
2	Normal(200, 50)	40	26	20	13	10	2	30	10
3	Normal(200, 50)	40	26	20	19	10	2	5	10
4	Normal(200, 50)	40	26	20	19	10	2	30	10
5	Uniform(100, 300)	40	26	20	13	10	2	5	10
6	Uniform(100, 300)	40	26	20	13	10	2	30	10
7	Uniform(100, 300)	40	26	20	19	10	2	5	10
8	Uniform(100, 300)	40	26	20	19	10	2	30	10

Table 8: Input parameters for transshipment with quantity discount model examples

Case	**Q1**	**Q2**	x^{eq}	**Supplier Profit**	**Retailer 1 Profit**	**Retailer 2 Profit**
1	221.001	221.001	279.254	4420.02	3944.86	3944.86
1	260.668	260.668	206.002	4448.04	4153.8	4153.8
1	206.002	279.254	252.68	4666.54	3976.37	3976.37
2	216.109	216.109	269.832	4322.18	3741.07	3741.07
2	261.331	261.331	214.029	4564.39	4081.51	4081.51
2	214.029	269.832	256.45	4744.94	3883.48	3883.48
3	221.001	221.001	226.878	4420.02	3944.86	3944.86
3	225.144	225.144	219.31	4491.21	3950.82	3950.82
3	219.31	226.878	235.468	No Stable C point		
4	216.109	216.109	221.513	4322.18	3741.07	3741.07
4	221.845	221.845	216.66	4426.53	3798.91	3798.91
4	216.66	221.513	247.637	No Stable C point		
5	222.662	222.662	286.233	4453.24	3489.06	3489.06
5	269.778	269.778	205.553	4496.41	3732.05	3732.05
5	205.553	286.233	267.956	4789.02	3495.15	3495.15

6	208.794	208.794	271.853	4175.88	3137.06	3137.06
6	266.94	266.94	214.725	4607.79	3655.71	3655.71
6	214.725	271.853	277.696	No Stable C point		
7	222.662	222.662	231.122	4453.24	3489.06	3489.06
7	228.378	228.378	219.925	4550.65	3496.95	3496.95
7	219.925	231.122	251.988	No Stable C point		
8	208.794	208.794	215.945	4175.88	3137.06	3137.06
8	216.098	216.098	209.078	4307.92	3239.9	3239.9
8	209.078	215.945	276.294	No Stable C point		

Table 9: Transshipment with quantity discount model example results. For each case, the first line is the pure newsvendor, the second line is the newsvendor with transshipment, and the third line is the newsvendor with transshipment and a quantity discount.

Notice from Table 8 and Table 9 that changing the quantity discount costs significantly affects the existence of the C equilibrium. Further examining of this event is shown in Table 10 and Table 11. In Table 10, c_r is varied and in Table 11 c_n is varied demonstrating that the C equilibrium does not always exist for small differences between c_n and c_r. Once the difference between c_n and c_r is large enough for the C equilibrium to exist, it will remain in existence for all larger differences between c_n and c_r that are valid inputs for the model.

Case	cr	Q1	Q2	x^{eq}	Supplier Profit	Retailer 1 Profit	Retailer 2 Profit	Total Profit
1 and 3	11	196.049	334.375	264.932	4679.25	4028.88	4028.88	12737.01
1 and 3	12	202.011	297.93	257.515	4676.09	3998.11	3998.11	12672.31
1 and 3	13	206.002	279.254	252.68	4666.54	3976.37	3976.37	12619.28
1 and 3	14	209.045	266.256	248.897	4648.86	3961.21	3961.21	12571.28
1 and 3	15	211.559	256.04	245.682	4624.2	3950.84	3950.84	12525.88
1 and 3	16	213.747	247.457	242.822	4593.5	3944.22	3944.22	12481.94
1 and 3	17	215.725	239.933	240.208	No Stable C Point			
1 and 3	18	217.563	233.142	237.772	No Stable C Point			
1 and 3	19	219.31	226.878	235.468	No Stable C Point			

Table 10: Cases 1 and 3 results shown with varying values of c_r. c_n = 20

Case	cn	Q1	Q2	x^{eq}	Supplier Profit	Retailer 1 Profit	Retailer 2 Profit	Total Profit
1 and 3	12	269.302	298.88	299.187	No Stable C Point			
1 and 3	13	254.531	302.434	291.291	1560.45	5579.41	5579.41	12719.27
1 and 3	14	243.37	306.117	285.352	2135.65	5330.32	5330.32	12796.29
1 and 3	15	233.99	310.008	280.533	2602.09	5091.91	5091.91	12785.91
1 and 3	16	225.629	314.163	276.462	3050.25	4862.86	4862.86	12775.97
1 and 3	17	217.881	318.63	272.952	3481.51	4642.37	4642.37	12766.25
1 and 3	18	210.488	323.455	269.893	3896.61	4430.01	4430.01	12756.63
1 and 3	19	203.263	328.685	267.228	4295.88	4225.53	4225.53	12746.94
1 and 3	20	196.049	334.375	264.932	4679.25	4028.88	4028.88	12737.01
1 and 3	21	188.696	340.593	263.008	5046.33	3840.13	3840.13	12726.59
1 and 3	22	181.033	347.435	261.493	5396.25	3659.53	3659.53	12715.31
1 and 3	23	172.841	355.057	260.466	5727.58	3487.59	3487.59	12702.76

Table 11: Cases 1 and 3 results shown with varying values of c_n. c_r = 11

Figure 26 and Figure 27 show a graphical representation of the data given in Table 10.

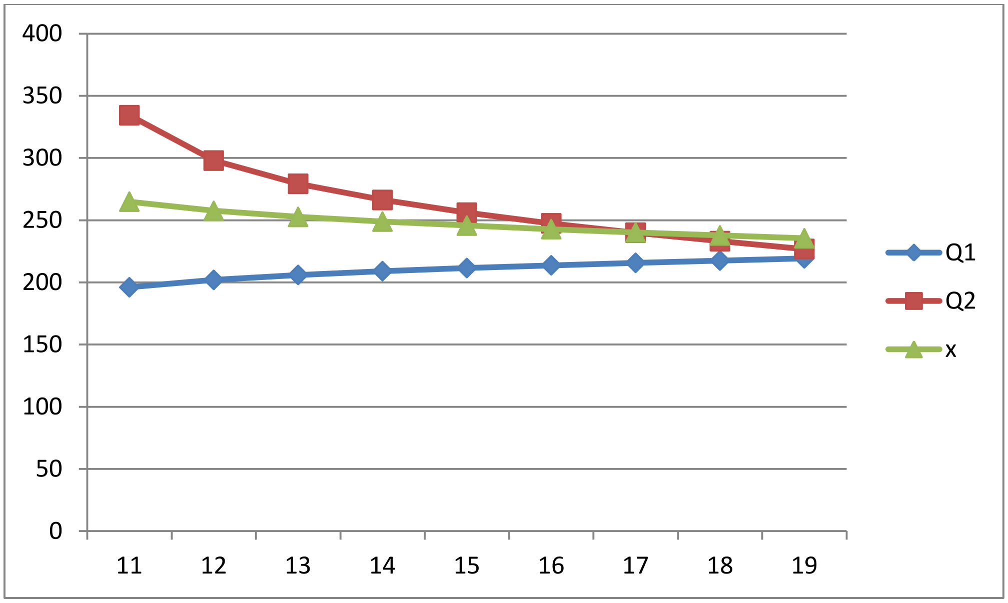

Figure 26: Q1, Q2, and x^{eq} for cases 1 and 3 as c_r changes, c_n= 20

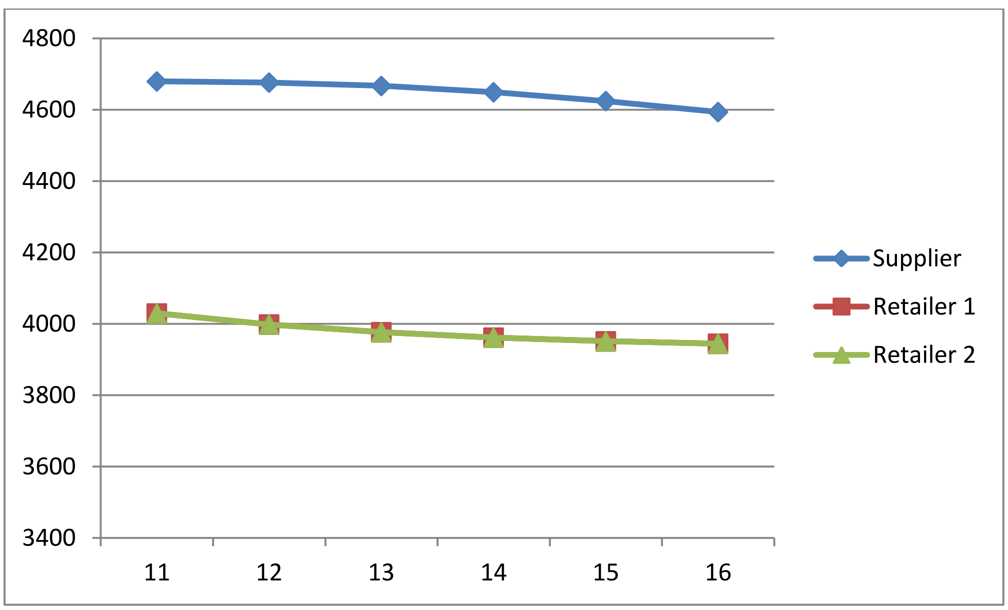

Figure 27: Retailer and Supplier profits for cases 1 and 3 as c_r changes, c_n= 20

Figure 28 and Figure 29 show a graphical representation of the data given in Table 11.

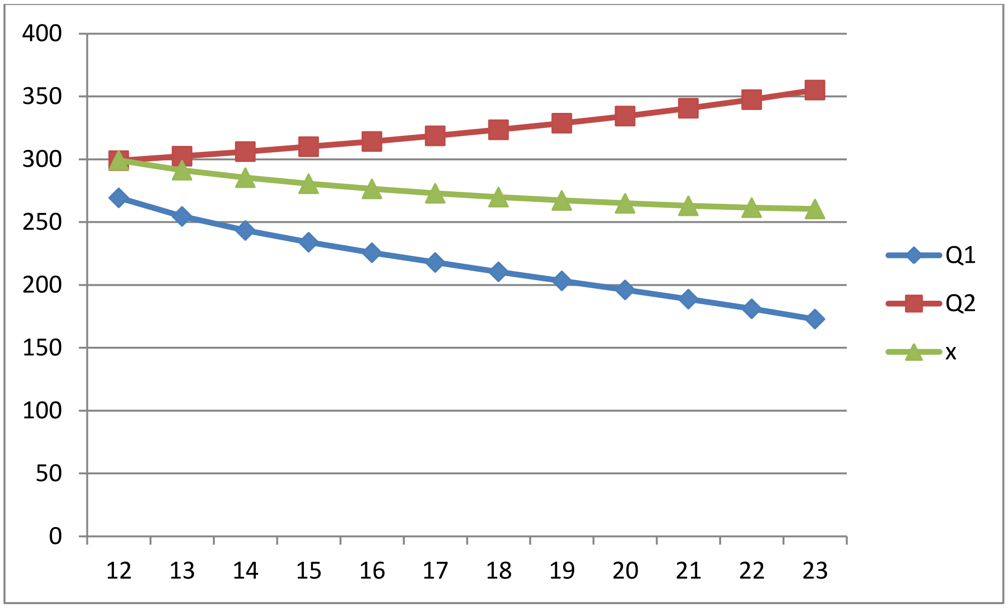

Figure 28: Q1, Q2, and $\underline{x}^{eq}$ for cases 1 and 3 as c_n changes, c_r= 11

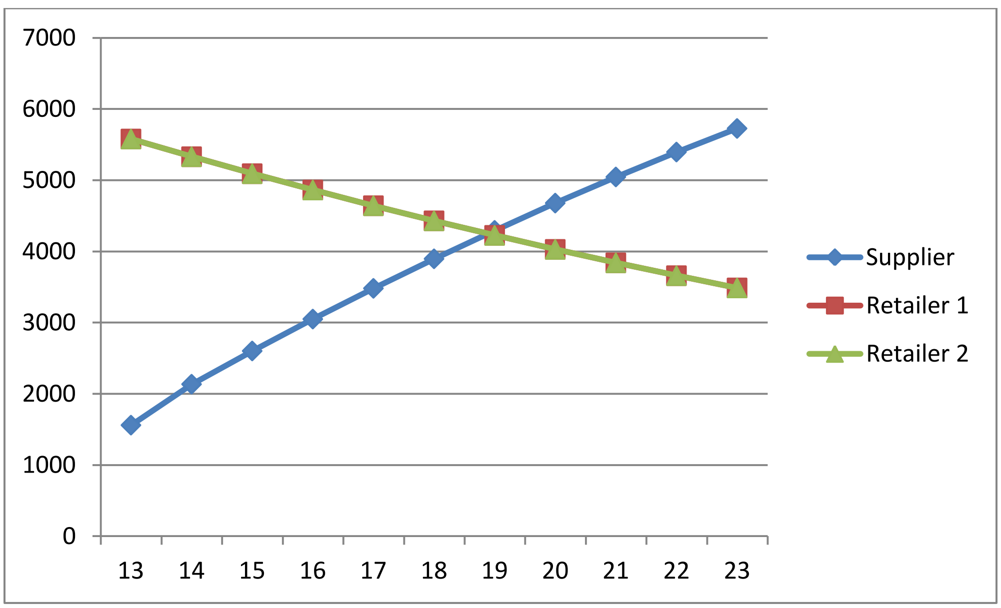

Figure 29: Retailer and Supplier profits for cases 1 and 3 as c_n changes, c_r= 11

In Table 10, as well as in Figure 27, we see that the supplier's profit, in addition to each retailer's profit and subsequently the total profit, increases as c_r decreases until the value of c_r is less than s, making the parameters invalid for the model. As seen in

Figure 26, no stable C point exists when c_r approaches the value of c_n, which in this case is the range of c_r = 17 through c_r = 19.

Additionally, Table 11 and Figure 29 show that the supplier's profit increases as c_n is increased. No stable C point exists for small differences between c_n and c_r, which in this case is when c_n = 12, which can be seen in Figure 28. From these results it appears that in order to receive the maximum profit, the supplier should set c_r to the lowest feasible value and set c_n to the highest feasible value while staying within the model limits, including $c_r > s$. These results help support the practical validity of the conditions from Proposition 2, which requires that the difference between c_n and c_r be larger, in order to ensure the existence of the C1 and C2 equilibria.

Chapter 4: Three Retailer Transshipment with a Central Retailer

Another possible point of interest is the scenario involving a third retailer where one retailer can transship with the other two retailers; however, the other two retailers cannot transship with each other. Figure 30 shows a flow diagram of the transshipment in the described system.

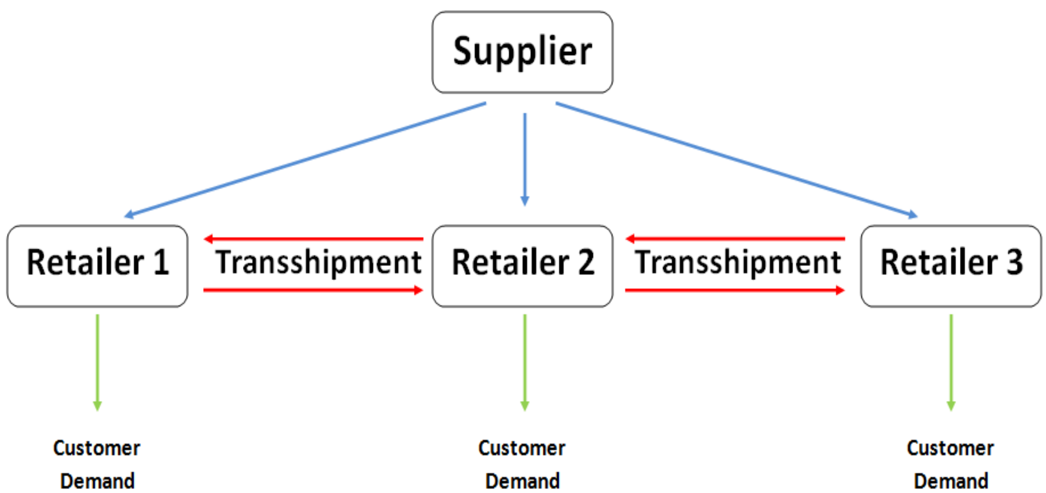

Figure 30: Flow Diagram of Proposed Scenario

This system is used in determining the value to each entity in the system from the previous chapter of one retailer adding an additional transshipment partner. In Chapter 3 it is shown that quantity discount pricing to the retailers from the supplier in addition to a transshipment agreement between the two retailers increases profit for both retailers and the supplier when compared to the newsvendor solution and the transshipment without quantity discount systems. This chapter builds an appropriate model representing the system described by Figure 30 and examines the retailer and supplier profits for comparison with the two retailer transshipment model with quantity discounts.

4.1: The Model

Section 4.1 encompasses many subsections that describe the three retailer transshipment with a central retailer model. Subsection 4.1.1 defines the essential notation used in this chapter. Subsection 4.1.2 defines the transshipment logic used for determining how transshipments are carried out within the transshipment system. Subsection 4.1.3 gives the model that is used as a basis for the three retailer transshipment model. In Subsection 4.1.4 the expected central retailer profit function is derived. In Subsection 4.1.5 the expected non-central retailer profit function is derived. Subsection 4.1.6 presents the transshipment event graphs associated with the three retailer transshipment model. In Subsection 4.1.7 the central retailer response function is defined. Finally, in Subsection 4.1.8 the non-central retailer response function is defined.

4.1.1: Essential Notation

In this subsection the essential notation used in the three retailer transshipment system is described. Let the subscript *c* be used to represent the central retailer in the three retailer system described here and let the subscript *nc,i* and *nc,j* be used to represent the two non-central retailers. Let *D* denote the customer demand, such that the demand for the central retailer is represented by D_c and the demand for the non-central retailers is represented by $D_{nc,i}$ and $D_{nc,j}$. Similarly, let Q denote the ordering quantity, such that the ordering quantity for the central retailer is represented by Q_c and the ordering quantity for the non-central retailers is represented by $Q_{nc,i}$ and $Q_{nc,j}$.

The amount transshipped from one retailer to another is denoted by T and uses the subscripts c and i to denote the direction of the transshipment. The first subscript represents the retailer sending the transshipment and the second subscript represents the retailer receiving the transshipment. Thus, T_{ic} represents the amount transshipped from one of the non-central retailers, i to the central retailer, c. Likewise, T_{ci} represents the amount transshipped from the central retailer, c, to a non-central retailer, i.

4.1.2: Transshipment Logic

In order to define the expected profit for each of the three retailers, transshipment rules must first be defined. The transshipment rules are divided into two categories, the first of which defines the transshipment rules when the central retailer has excess inventory while the second defines the transshipment rules when the central retailer has excess demand. In this system we assume the following rules for transshipment when the central retailer has excess inventory :

1 Initial Allocation of Excess Central Inventory: If the central retailer has excess inventory available after satisfying local demand and both non-central retailers have excess demand, the excess inventory of the central retailer is initially divided evenly and allocated for potential transshipment to each of the two non-central retailers. If the central retailer has excess inventory available after satisfying local demand and only one non-central retailer has excess demand, then all of the excess inventory from the central retailer is allocated to the non-central retailer with excess demand.

2 Final Reconciliation of Excess Central Inventory: Based on the Initial Allocation of Excess Central Inventory, if the excess demand at one of the

non-central retailers is more than the amount initially allocated to them (the "short retailer") while the excess demand at the other retailer is less the amount initially allocated to them (the "satisfied retailer"), then the excess inventory allocated for the satisfied retailer will be used to meet the demand at the short retailer.

Note that if the central retailer has excess inventory and both non-central retailers have no excess demand then transshipment does not occur.

We assume the following rules for transshipment when the central retailer has excess demand:

1 Initial Offering of Excess Central Demand: When the central retailer has excess demand after using local stock to satisfy local demand, it will accept (an) incoming transshipment(s). This excess demand of the central retailer is initially divided evenly and an equal amount is offered to each of the two non-central retailers for fulfillment. If the central retailer has excess demand available after satisfying local demand and only one non-central retailer has excess inventory, then all of the excess demand from the central retailer is allocated to be fulfilled by the non-central retailer with excess inventory.

2 Final Reconciliation of Excess Central Demand: When the central retailer has excess demand, if one of the non-central retailer's excess stock exceeds the demand initially offered to them (the "overstocked retailer") while the excess stock at the other non-central retailer is less than the

amount initially offered to them (the "stocked out retailer"), then the excess demand offered to the stocked out retailer will be offered to the overstocked retailer for fulfillment.

Note that if the central retailer has excess demand and both non-central retailers have no excess inventory then transshipment does not occur.

4.1.3: Model Origins

The basis for this model is the two retailer version of the transshipment with quantity discounts that is described in Equation 9, Equation 4, and Equation 10; where π_i^{qd} is the expected retailer profit for each retailer.

$$\pi_i^{qd} = \begin{cases} \pi_i^d, & Q_i \le x \\ \pi_i^r, & Q_i > x \end{cases} \tag{9}$$

$$\pi_i^d(Q_i, Q_j) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} - c_n Q_i \tag{4}$$

$$\pi_i^r(Q_i, Q_j) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} - x(c_n - c_r) - c_r Q_i. \tag{10}$$

4.1.4: Expected Central Retailer Profit

Let the subscript *c* be used to represent the central retailer in the three retailer system described here and let the subscript *nc,i* and *nc,j* be used to represent the two non-central retailers. Let π_c^{qd} represent the expected profit for the central retailer in the three retailer system. The expected profit for the central retailer is given in Equation 24.

$$\pi_c^{qd} = \begin{cases} \pi_c^d, & Q_c \le x \\ \pi_c^r, & Q_c > x \end{cases} \tag{24}$$

Where Q_c is the ordering quantity of the central retailer, π_c^d is defined as:

$$\pi_c^d\left(Q_c, Q_{nc,i}, Q_{nc,j}\right)$$
$$= E\left\{r_c R_c - c_{ic}T_{ic} - c_{jc}T_{jc} + (c_{ci} - \tau_{ci})T_{ci} + \left(c_{cj} - \tau_{cj}\right)T_{cj} + s_c U_c - p_c Z_c\right\} - c_n Q_c \quad (25)$$

$Q_{nc,i}$ and $Q_{nc,j}$ are the ordering quantities of the two non-central retailers, and π_c^r is defined as:

$$\pi_c^r\left(Q_c, Q_{nc,i}, Q_{nc,j}\right)$$
$$= E\left\{r_c R_c - c_{ic}T_{ic} - c_{jc}T_{jc} + (c_{ci} - \tau_{ci})T_{ci} + \left(c_{cj} - \tau_{cj}\right)T_{cj} + s_c U_c - p_c Z_c\right\} - x(c_n - c_r) - c_r Q_c \quad (26)$$

The amount of inventory to be transshipped from a non-central retailer, retailer *i*, to the central retailer, retailer *c*, is represented as T_{ic}, and is defined as:

$$T_{ic} = \min\left[\max\left(Q_{nc,i} - D_{nc,i}\right)^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max\left(Q_{nc,j} - D_{nc,j}\right)^+\right)^+\right]^+\right] \quad (27)$$

The amount of inventory to be transshipped from the central retailer, retailer *c* to a non-central retailer, retailer *i*, is represented as T_{ci}, and is defined as:

$$T_{ci} = \min\left[\max\left(D_{nc,i} - Q_{nc,i}\right)^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max\left(D_{nc,j} - Q_{nc,j}\right)^+\right)^+\right]^+\right] \quad (28)$$

Retail sales for retailer *c* are defined as:

$$R_c = \min(D_c, Q_c) + T_{ic} + T_{jc} \quad (29)$$

Unsold stock for retailer *c* is defined as:

$$U_c = \max\left(Q_c - D_c - T_{ci} - T_{cj}\right)^+ \quad (30)$$

And unmet demand for retailer *c* is defined as:

$$Z_c = \max\left(D_c - Q_c - T_{ic} - T_{jc}\right)^+ \tag{31}$$

Combining Equations 27, 28, 29, 30, and 31 with Equation 26 results in the expanded version of π_c^r, which is given in Appendix A. Similarly, those equations can be combined with Equation 25 to get the expanded version of π_c^d, the only difference being the cost per unit from the supplier. The expanded version of π_c^d can also be found in Appendix A

Expanding out the expectation operator and rearranging the terms results in the integral form of π_c^r, which is given in Appendix B. The integral form of π_c^d is similar to π_c^r, the only difference being the cost per unit from the supplier. The integral form of π_c^d can also be found in Appendix B.

4.1.5: Expected Non-Central Retailer Profit

Let $\pi_{nc,i}^{qd}$ represent the expected profit for one of the two non-central retailers.

$$\pi_{nc,i}^{qd} = \begin{cases} \pi_{nc,i}^{d}, & Q_{nc,i} \le x \\ \pi_{nc,i}^{r}, & Q_{nc,i} > x \end{cases} \tag{32}$$

$\pi_{nc,i}^{d}$ is defined as:

$$\pi_{nc,i}^{d}\left(Q_c, Q_{nc,i}, Q_{nc,j}\right) \\ = E\left\{r_{nc,i}R_{nc,i} - c_{ci}T_{ci} + (c_{ic} - \tau_{ic})T_{ic} + s_{nc,i}U_{nc,i} - p_{nc,i}Z_{nc,i}\right\} - c_n Q_{nc,i} \tag{33}$$

and $\pi_{nc,i}^{r}$ is defined as:

$$\pi_{nc,i}^{r}\left(Q_c, Q_{nc,i}, Q_{nc,j}\right) = E\left\{r_{nc,i}R_{nc,i} - c_{ci}T_{ci} + (c_{ic} - \tau_{ic})T_{ic} + s_{nc,i}U_{nc,i} - \\ p_{nc,i}Z_{nc,i}\right\} - x(c_n - c_r) - c_r Q_{nc,i} \tag{34}$$

Retail sales for retailer *i* are defined as:

$$R_{nc,i} = \min\left(D_{nc,i}, Q_{nc,i}\right) + T_{ci} \tag{35}$$

Unsold stock for retailer *c* is defined as:

$$U_{nc,i} = \max\left(Q_{nc,i} - D_{nc,i} - T_{ic}\right)^{+} \tag{36}$$

And unmet demand for retailer *c* is defined as:

$$Z_{nc,i} = \max\left(D_{nc,i} - Q_{nc,i} - T_{ci}\right)^{+} \tag{37}$$

Combining Equations 27, 28, 35, 36, and 37 with Equation 34 results in the expanded version of $\pi^{r}_{nc,i}$, which is given in Appendix C. Similarly, those equations can be combined with Equation 33 to get the expanded version of $\pi^{d}_{nc,i}$, the only difference being the cost per unit from the supplier. The expanded version of $\pi^{d}_{nc,i}$ can also be found in Appendix C.

Expanding out the expectation operator and rearranging the terms results in the integral form of $\pi^{r}_{nc,i}$, which is given in Appendix D. The integral form of $\pi^{d}_{nc,i}$ is similar to $\pi^{r}_{nc,i}$, the only difference being the cost per unit from the supplier. The integral form of $\pi^{d}_{nc,i}$ can also be found in Appendix D.

4.1.6: Event Diagrams

Unlike the two retailer transshipment scenario, which can be described by events in a two dimensional demand space, the three retailer scenario produces transshipment events that must consider a three dimensional demand space. The simplest regions within the three retailer transshipment event space are the zero transshipment areas, which are shown in Figure 31 and Figure 32. The upper region in Figure 31 represents the area where the demand for each retailer is larger than that retailer's ordering quantity. The lower region in Figure 31 represents the area where the demand for each retailer is smaller than that retailer's ordering quantity. In both regions, no transshipments occur.

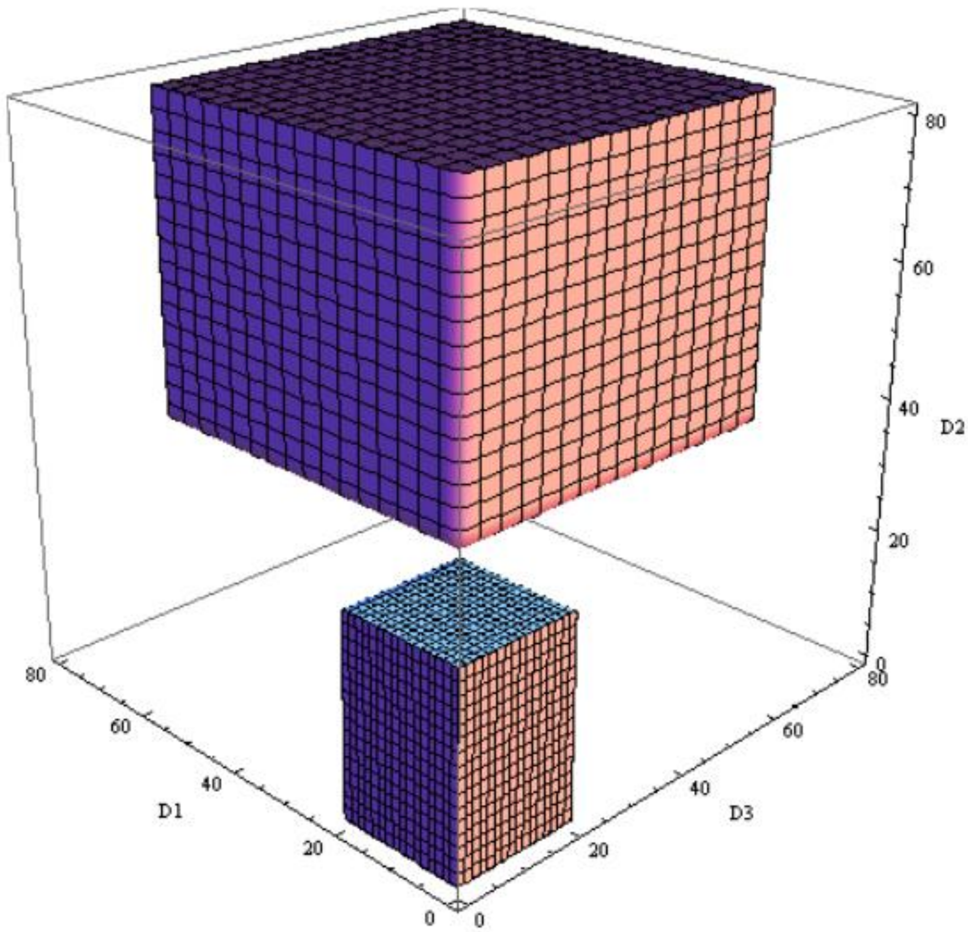

Figure 31: Zero transshipment areas of the three retailer transshipment with quantity discounts case event graph.

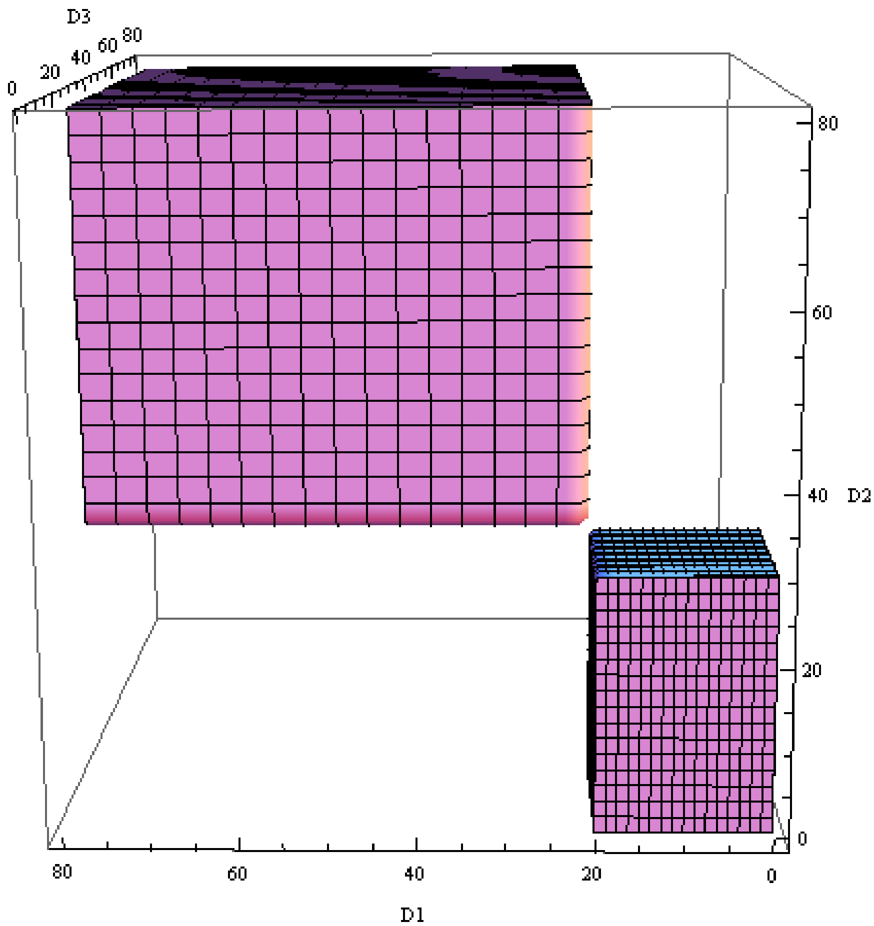

Figure 32: Zero transshipment areas of the three retailer transshipment with quantity discounts case event graph.

The diagram representing regions of demand where transshipment from retailer 1 to retailer 2 occurs is shown in Figure 33.

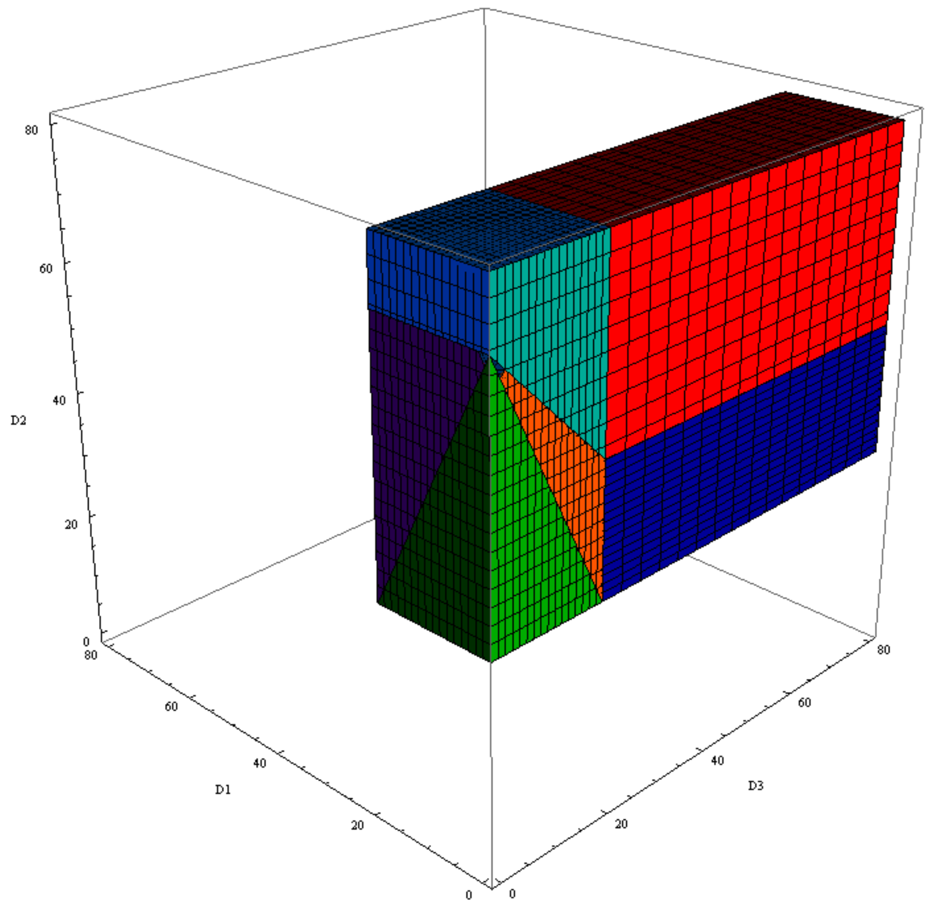

Figure 33: Regions representing the transshipment from retailer 1 to retailer 2 of the three retailer transshipment with quantity discounts case event graph

Note that all of the transshipment events occur while the demand for retailer 1 is low while the demand for retailer 2 is high. The transshipment events here occur close to the d1 = 0 axis and no transshipments occur once the demand for retailer 1 is higher than retailer 1's ordering quantity. The demand for retailer 2 must also be higher than retailer 2's ordering quantity for these transshipment events to occur. A similar diagram can be generated for retailer 3's transshipment to retailer 2. The event diagram for transshipment from retailer 3 to retailer 2 is shown in Figure 34.

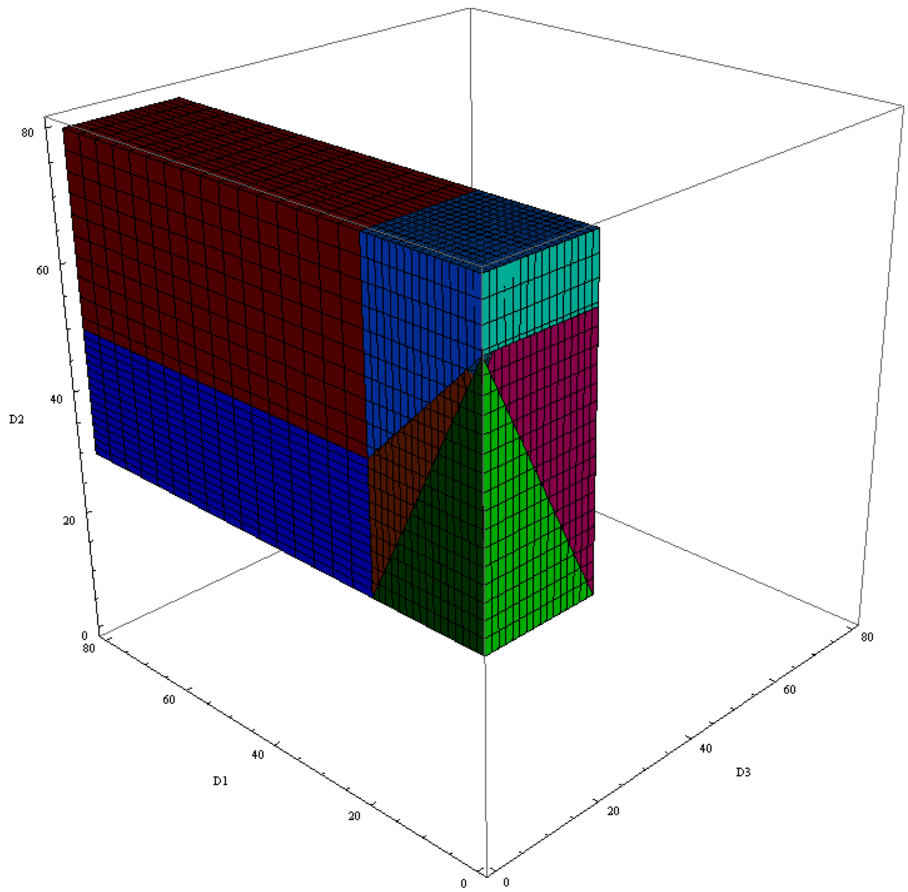

Figure 34: Regions representing the transshipment from retailer 3 to retailer 2 of the three retailer transshipment with quantity discounts case event graph

Note that there is overlap in the transshipment events shown in Figure 33 and Figure 34 near the d2 axis (where d1 = d3 = 0). This is because in this region, both retailers transship to retailer 2. The overlapping volume is comprised of several regions, where each region represents a different amount of inventory transshipped to retailer 2 from retailer 1 and from retailer 3 that is based upon the transshipment rules described in Section 4.1.2. Similar event diagrams represent the transshipment from retailer 2 to retailers 1 and 3. Figure 35 shows the transshipment events when retailer 2 transships to retailer 1.

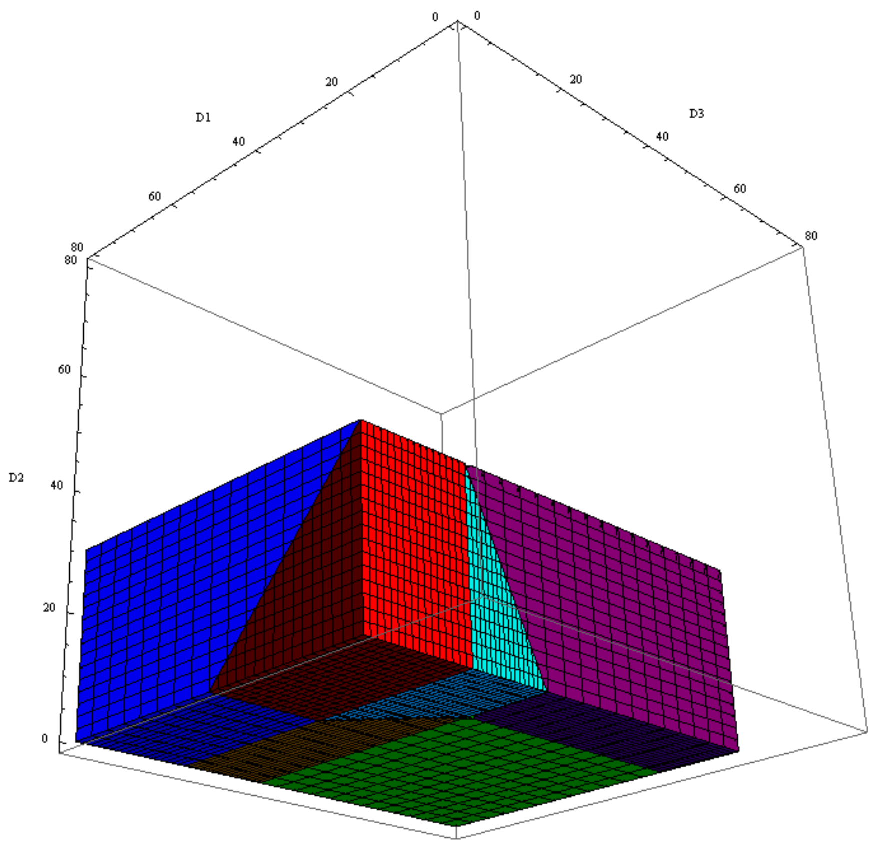

Figure 35: Regions representing the transshipment from retailer 2 to retailer 1 of the three retailer transshipment with quantity discounts case event graph

Note that transshipments from retailer 2 to retailer 1 only occur when there is an excess of inventory for retailer 2 while there is an excess in demand for retailer 1. This can be seen in Figure 35 where the transshipment events occur for low values of demand for retailer 2 and higher values of demand for retailer 1. Similarly, Figure 36 shows the transshipment events when retailer 2 transships to retailer 3.

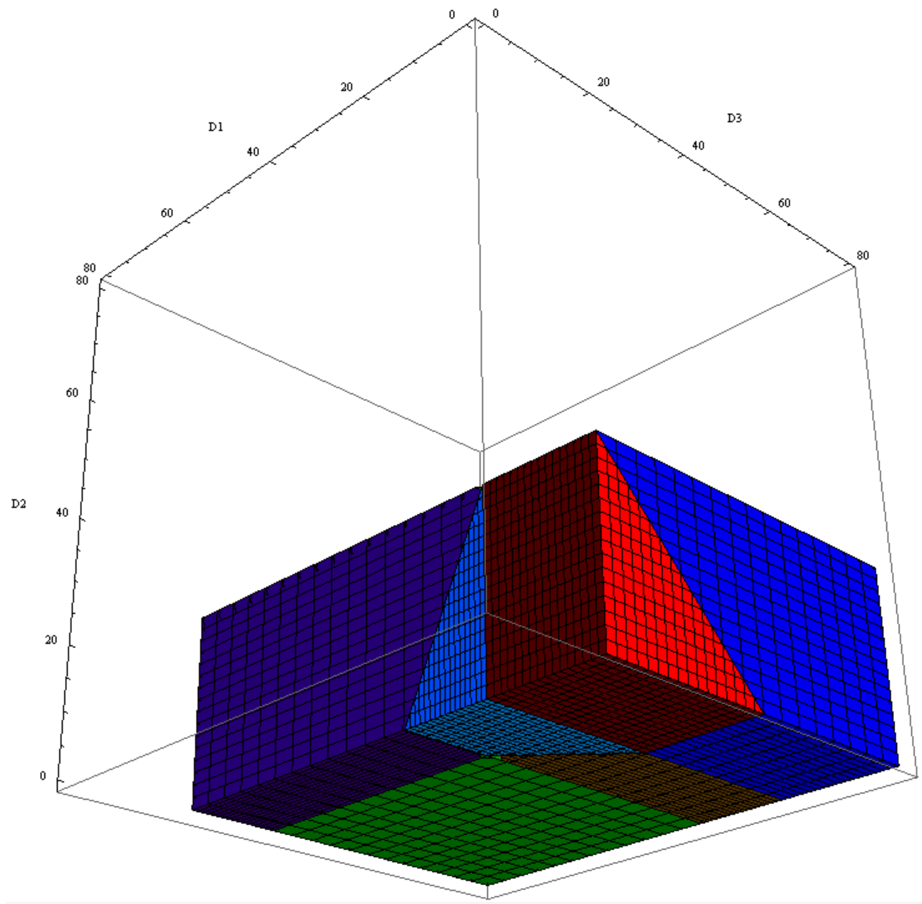

Figure 36: Regions representing the transshipment from retailer 2 to retailer 3 of the three retailer transshipment with quantity discounts case event graph

Note that Figure 35 and Figure 36 share some overlapping regions where transshipments occur from retailer 2 to both retailer 1 and retailer 3. A further explanation of the transshipment event diagram can be found in Appendix E.

The construction of the event diagrams not only gives a graphical depiction of the transshipment events, but also helped in verifying the derivation of the profit function used in this scenario. Additionally, as an additional verification, the probability density functions (pdfs) of demand were integrated numerically in the three dimensions over the different regions of the event diagrams shown here. As they must, these sums of integrals were shown to sum to one. Thus, this verifies that all possible demand realizations in the demand space are accounted for in the event diagrams. It also helped verify the integral formulations (in particular the limits of integration) used in the profit functions. The formula showing the summation of the pdfs to one can be found in Appendix F.

4.1.7: Central Retailer Response Function

Next we define $k_c(Q_c, Q_{nc,i}, Q_{nc,j})$, which is given in Appendix G. Differentiation of π_c^r with respect to Q_c results in the following equation:

$$\frac{\partial \pi_c^r}{\partial Q_c} = k_c(Q_c, Q_{nc,i}, Q_{nc,j}) - c_r \tag{38}$$

A similar result is derived from π_c^d when differentiation with respect to Q_c is applied as shown in the following equation:

$$\frac{\partial \pi_c^d}{\partial Q_c} = k_c(Q_c, Q_{nc,i}, Q_{nc,j}) - c_n \tag{39}$$

Setting the derivatives in Equation 38 and Equation 39 equal to zero and solving for Q_c as a function of $Q_{nc,i}$ and $Q_{nc,j}$ results in the optimal ordering quantity for the central retailer, Q_c^*, as a function of the other retailers' ordering quantities. Thus Equation 40 and Equation 41 define the response functions for the central retailer.

$$k_c(Q_c^*, Q_{nc,i}^*, Q_{nc,j}^*) = c_r \tag{40}$$

$$k_c(Q_c^*, Q_{nc,i}^*, Q_{nc,j}^*) = c_n \tag{41}$$

4.1.8: Non-Central Retailer Response Function

Similarly, we define $k_{nc,i}(Q_c, Q_{nc,i}, Q_{nc,j})$, which is given in Appendix H. Differentiation of $\pi_{nc,i}^r$ with respect to $Q_{nc,i}$ results in the following equation:

$$\frac{\partial \pi_{nc,i}^r}{\partial Q_{nc,i}} = k_{nc,i}(Q_c, Q_{nc,i}, Q_{nc,j}) - c_r \tag{42}$$

A similar result is derived from $\pi^d_{nc,i}$ when differentiation with respect to $Q_{nc.i}$ is applied as shown in the following equation:

$$\frac{\partial \pi^d_{nc,i}}{\partial Q_{nc,i}} = k_{nc,i}\left(Q_c, Q_{nc,i}, Q_{nc,j}\right) - c_n \tag{43}$$

Setting the derivatives in Equation 42 and Equation 43 equal to zero and solving for $Q_{nc,i}$ as a function of Q_c and $Q_{nc,j}$; and $Q_{nc,j}$ as a function of Q_c and $Q_{nc,i}$ results in the optimal ordering quantities for retailer i and retailer j ($Q^*_{nc,i}, Q^*_{nc,j}$) as a function of the other retailers' ordering quantities. Thus Equation 44 and Equation 45 define the response functions for retailer i.

$$k_{nc,i}\left(Q^*_c, Q^*_{nc,i}, Q^*_{nc,j}\right) = c_r \tag{44}$$

$$k_{nc,i}\left(Q^*_c, Q^*_{nc,i}, Q^*_{nc,j}\right) = c_n \tag{45}$$

4.2: Expected Supplier Profit

The expected supplier profit π^t, is given by Equation 46, where c_s is the supplier's production cost.

$$\pi^t = \left(Q_c + Q_{nc,i} + Q_{nc,j}\right) * c_r + (c_n - c_r) * \left[\mathrm{Max}(Q_c, x) + \mathrm{Max}\left(Q_{nc,i}, x\right) + \mathrm{Max}\left(Q_{nc,j}, x\right)\right] - \left(Q_c + Q_{nc,i} + Q_{nc,j}\right) * c_s \tag{46}$$

4.3: Results

The results section of the three retailer transshipment with quantity discounts case first analyzes the response functions of the retailers. From the response functions, the potential equilibria can be identified and from there the stable regions of the discount triggering quantity can be identified. The expected profits of both the supplier and the three retailers are calculated and examined with respect to varying costs from the supplier. Lastly, a comparison is made between the three retailer case and the two retailer case such that conclusions can be made about the value added to the various involved parties of introducing a third retailer into the transshipment system in accordance to the method previously described in this chapter.

Unless otherwise stated, an exponential demand distribution is used in this section for computational results in order to reduce the time needed for generating results. In general the computational results can be obtained for any distribution function. In general, the following parameters are used for generating graphs and making comparisons: $c_{ic} = c_{ci} = c_{jc} = c_{cj} = 26$; $r_c = r_{nc,i} = r_{nc,j} = 40$; $s_c = s_{nc,i} = s_{nc,j} = 10$; $p_c = p_{nc,i} = p_{nc,j} = 5$; $\tau_{ic} = \tau_{ci} = \tau_{jc} = \tau_{cj} = 2$; $D_c, D_i, D_j \sim$ Expo (mean = 200). c_n and c_r are varied and the values of which are later determined by the supplier based upon expected supplier profit.

4.3.1: The Retailers' Response Functions

The response functions for the three retailers can be viewed in a three dimensional graph. In order to view the response functions of the three retailer transshipment with quantity discount system in a two dimensional response graph, it is assumed that the ordering quantities for the two non-central retailers are equal, $Q_{nc,i} = Q_{nc,j}$. Due to the assumption that $Q_{nc,i} = Q_{nc,j}$ the response function graph of Q_c and $Q_{nc,i}$ is identical to the response function graph of Q_c and $Q_{nc,j}$. Additionally, it is important to note that the response curves for the central retailer and a non-central retailer are not mirror images of each other, which was the case for the two retailers in the two retailer system. This is due to the fact the central retailer can transship with both non-central retailers, while each non-central retailer can only transship with the central retailer.

In the two retailer scenario, changing c_n and c_r does not affect the existence of the potential equilibria, other than if the difference between c_n and c_r is very small then the C1 and C2 equilibria do not exist. However, in the three retailer scenario, c_n and c_r have more of an impact on the existence of the equilibria. Figure 37 shows a single set of response curves when the cost per unit bought from the supplier is low.

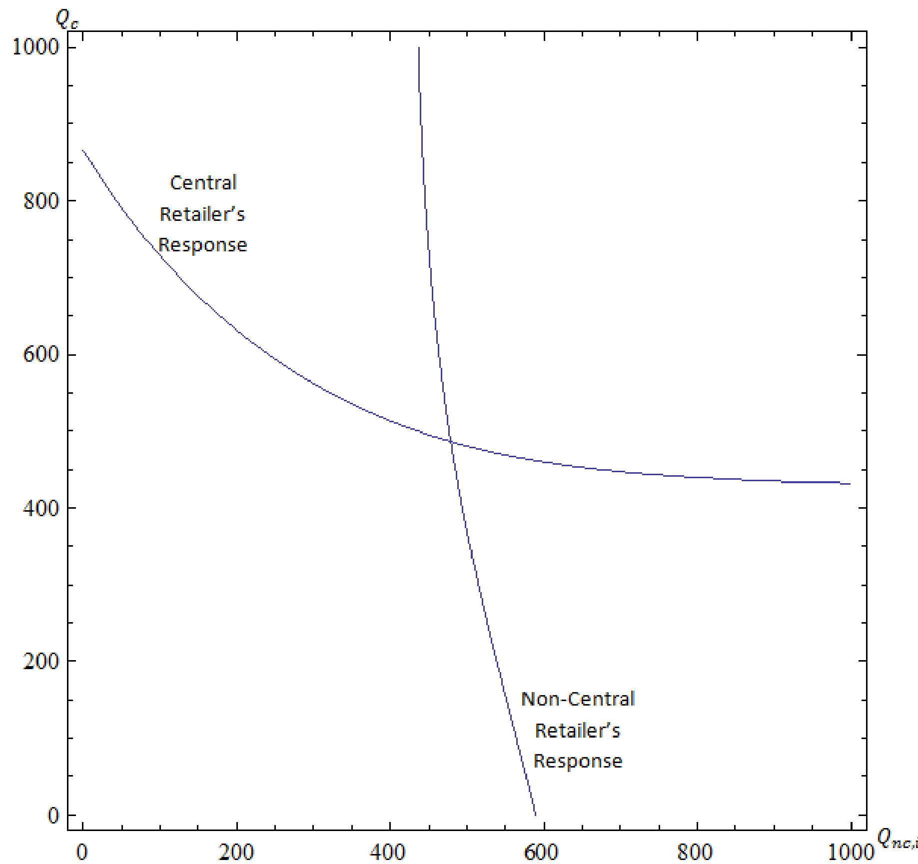

Figure 37: Response curves for Q_c and $Q_{nc,i}$; cost from supplier of 13 per unit.

As the cost per unit bought from the supplier increases, inflection points develop in both curves but at different rates. As the cost per unit bought from the supplier increases, the inflection point can first be noticed in the response curve of the central retailer, then as the cost per unit continues to increase, it is noticeable in the response curve of the non-central retailer. Figure 38 shows a single set of response curves when the cost per unit bought from the supplier is higher.

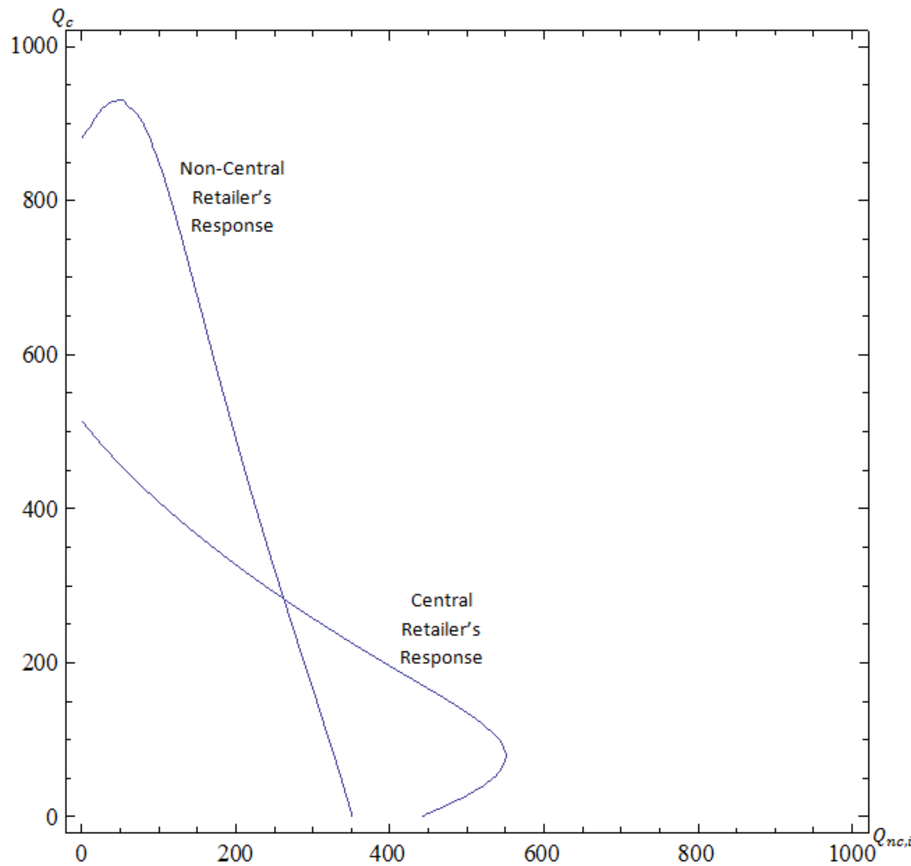

Figure 38: Response curves for Q_c and $Q_{nc,i}$; cost from supplier of 23 per unit.

Through different combinations of the values for c_n and c_r, the existence of the different equilibria can be controlled by the supplier. For larger differences between c_n and c_r, there exists only two equilibria, which are labeled as Equilibrium A and Equilibrium B. The response functions using this combination of c_n and c_r can be seen in Figure 39. As in the two retailer system, the intersection where both retailers are on the response curve where the normal cost from the supplier is used is identified as Equilibrium A, while the response curve where the reduced cost from the supplier is used is identified as Equilibrium B.

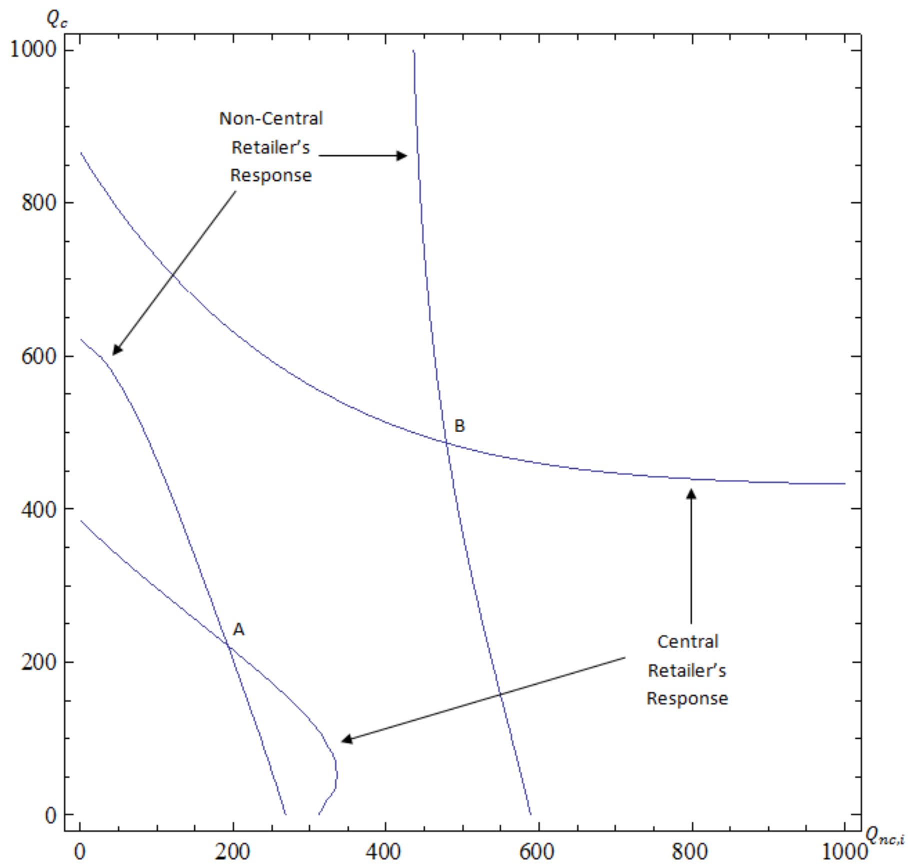

Figure 39: Response functions for Q_c and $Q_{nc,i}$, where c_n=25 and c_r=13.

It is possible to obtain a response function graph that resembles the response function graph from the two retailer system. This occurs when the value of c_n is small enough that the inflection points do not result in a second equilibrium between the crossing of a normal cost response curve and reduced cost response curve. A combination of c_n and c_r resulting in this category of response function graph is given in Figure 40. Equilibrium C1a and Equilibrium C2a represent equilibria that are a result of the intersection between a reduced cost response curve and a normal cost response curve, where on the response curve containing the inflection, the intersection is at a larger value of the ordering quantity than that of the point of inflection.

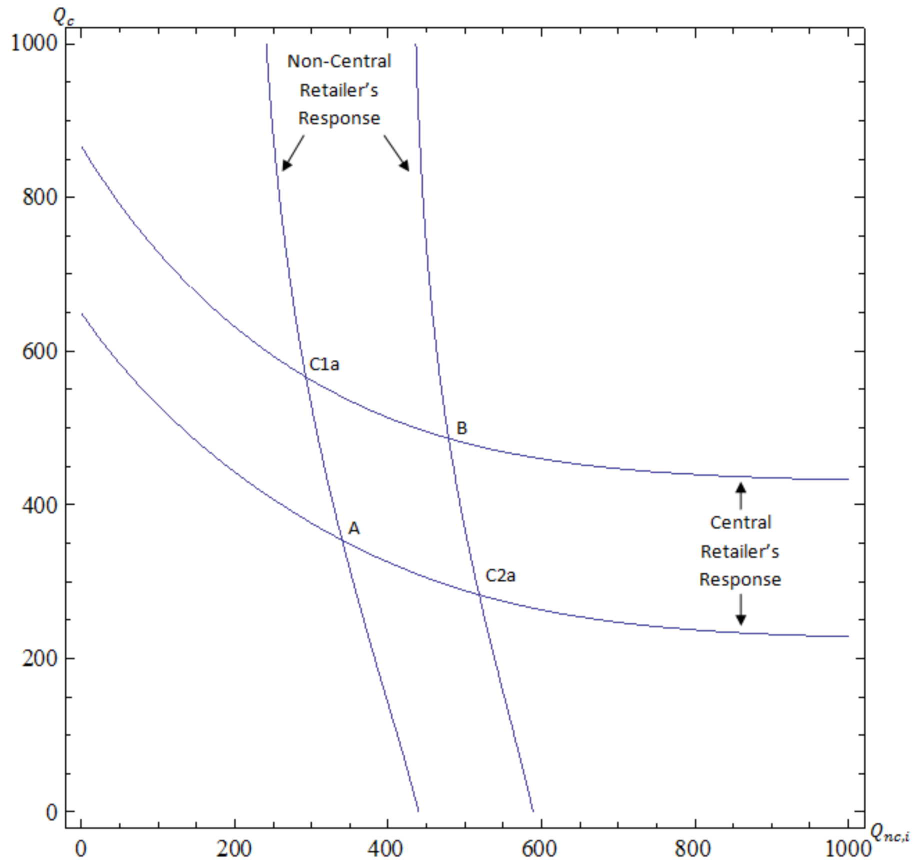

Figure 40: Response functions for Q_c and $Q_{nc,i}$, where c_n=16 and c_n=13.

As c_n increases, an inflection point first develops in the response curve for the central retailer for the normal cost per unit bought from the supplier. Thus for this case, carefully chosen combinations of c_n and c_r result in a second intersection between the reduced cost response curve of the non-central retailer and the normal cost response curve of the central retailer. This intersection is at a smaller value of Q_c than the value of Q_c at the inflection point on the normal cost response curve for the central retailer and is labeled as Equilibrium C2b. Figure 41 shows an example combination of c_n and c_r that results in this type of response function graph.

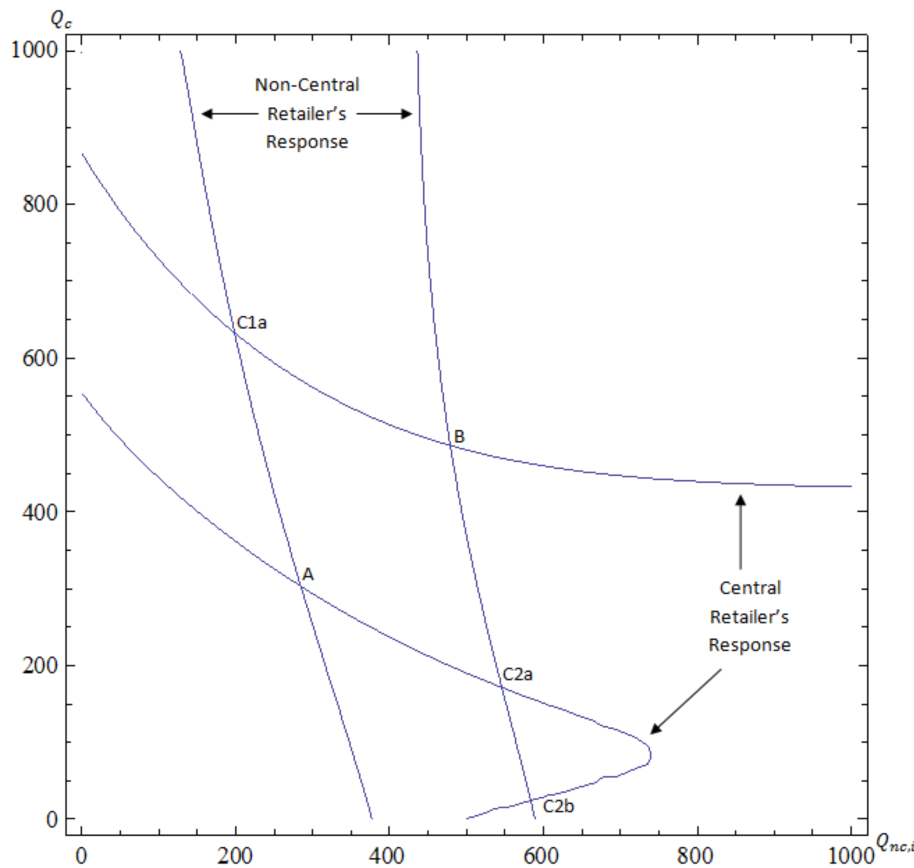

Figure 41: Response functions for Q_c and $Q_{nc,i}$, where c_n=18 and c_r=13.

As c_n continues to increase, an inflection point develops in the response curve for the non-central retailer when using a normal cost per unit bought from the supplier. While Equilibrium C2a and Equilibrium C2b no longer are in existence, it is possible to obtain response functions that result in the existence of both Equilibrium C1a and Equilibrium C1b. Equilibrium C1b identifies the intersection of a smaller value of $Q_{nc,i}$ or $Q_{nc,j}$ than the value of that ordering quantity at the inflection point on the normal cost response curve for the non-central retailer. Figure 42 shows an example combination of c_n and c_r that results in this type of response function graph.

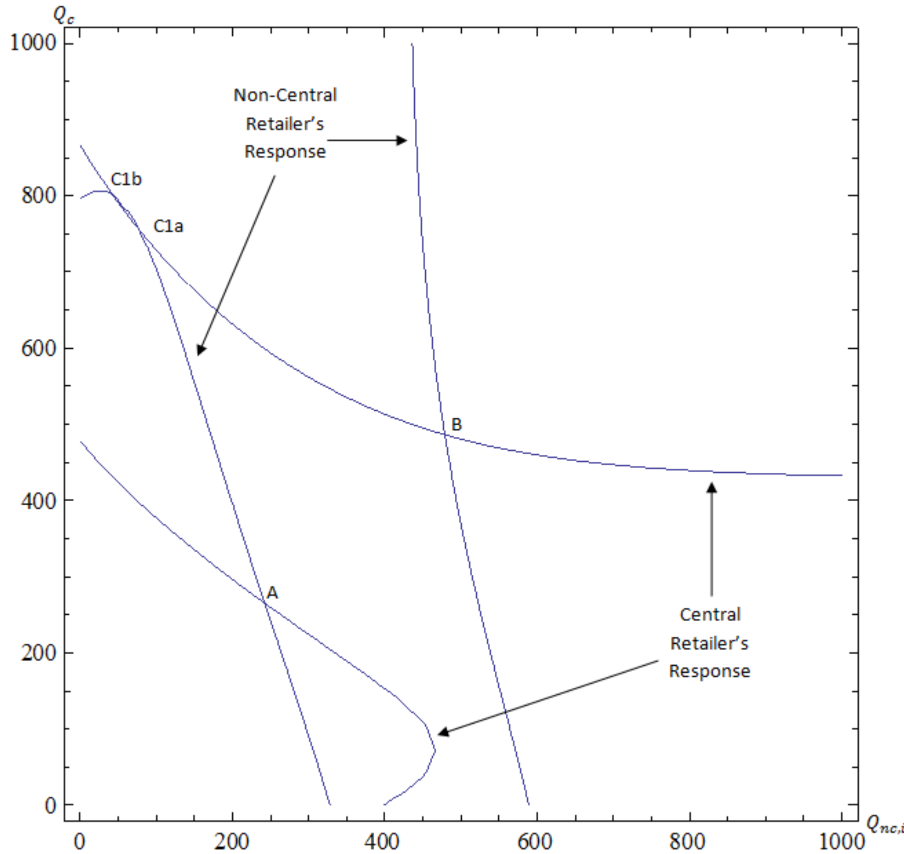

Figure 42: Response functions for Q_c and $Q_{nc,i}$, where c_n=20 and c_r=13.

4.3.2: Supplier Decisions

The supplier has control of several variables that affect the expected profits for both the supplier and the retailers. Included in these variables is the discount triggering quantity, the normal cost per unit bought from the supplier, and the reduced cost per unit bought from the supplier. Through the variation of the normal cost per unit bought from the supplier and the reduced cost per unit bought from the supplier, the supplier can affect which stable regions exist for the retailers. Through the variation of the discount triggering quantity, the supplier can control the retailer decisions regarding which equilibrium the retailers will order in accordance with.

In order to optimize the expected supplier profit and other measures of supplier performance, the supplier can start by analyzing how the variation of the discount triggering quantity affects the ordering quantities of the retailers. The supplier must consider the stability or retailer decisions for different values of the discount triggering quantity. We extend the definition of stability from the definition for the two retailer system so that it is suited to the three retailer system. Thus, in the three retailer system, we define:

> a region of the discount triggering quantity is defined to be *stable* if an equilibrium is in existence that results in the largest expected retailer profit for all involved retailers out of all possible equilibria in existence for that particular value of the discount triggering quantity.

An example graph of the expected retailer profit for a specific combination of values for c_n and c_r is shown in Figure 43. The legend for Figure 43 as well as the ordering quantities for each retailer at each of the equilibria is given in Table 12.

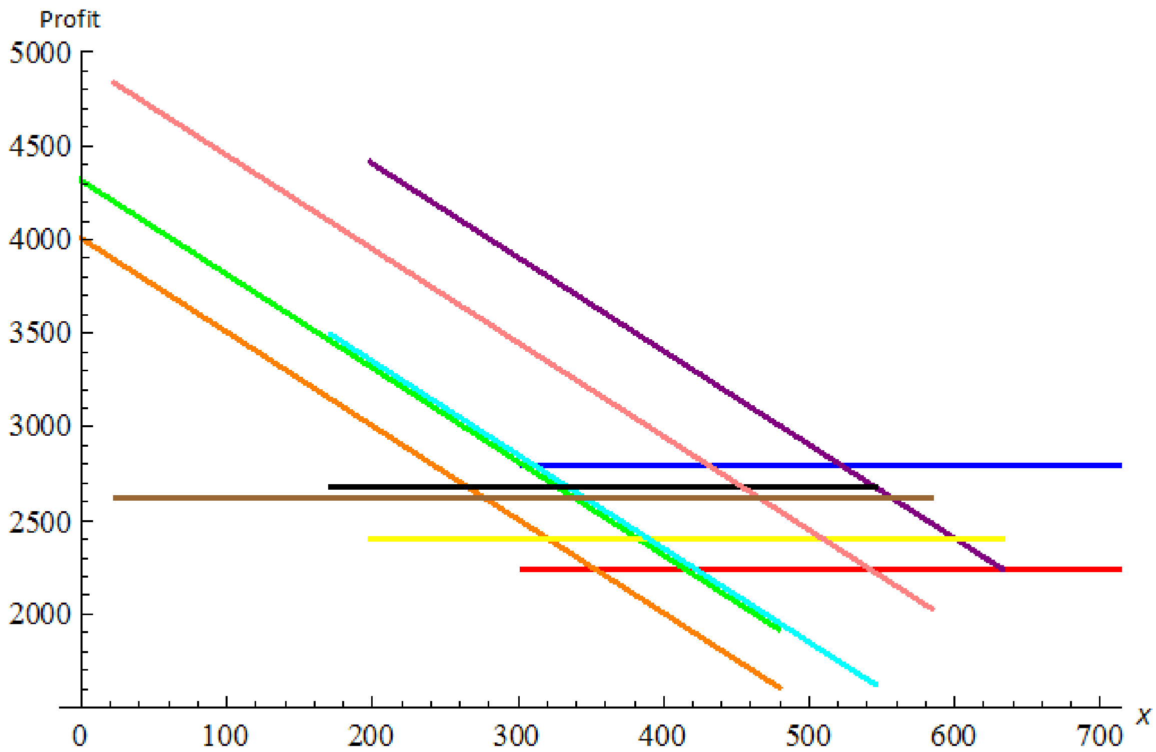

Figure 43: Expected retailer profit as *x* changes in the three retailer system where c_n=18 c_r=13.

Intersection	Central Retailer Profit ID	Central Retailer Ordering Quantity	Non-Central Retailer Profit ID	Non- Central Retailer Ordering Quantity
A		302.936		284.42
B		486.255		478.789
C1a		632.556		198.459
C2a		171.596		545.710
C2b		23.404		583.409

Table 12: Legend for Figure 43.

From Figure 43 the supplier can determine the stable regions of the discount triggering quantity for this particular combination of c_n and c_r. Table 13 shows the stable regions of the discount triggering quantity that pertains to Figure 43.

Retailer's Choice		
x	**Choice**	**Supplier's Profit Trend**
0 to $Q^*_{c,C2b}$	B	Increasing as x increases
$Q^*_{c,C2b}$ to $x^{eq,Q_{nc}}_{C2b,C1a}$	Unstable	Unstable
$x^{eq,Q_{nc}}_{C2b,C1a}$ to $x^{eq,Q_c}_{C1a,A}$	C1a	Increasing as x increases
$x^{eq,Q_c}_{C1a,A}$ to $Q^*_{c,C1a}$	Unstable	Unstable
$Q^*_{c,C1a}$ to ∞	A	Constant as x increases

Table 13: Retailer's choice in equilibrium and the supplier's expected profit trend given the value of *x*.

Here:

$Q^*_{c,C2b}$ is the ordering quantity of the central retailer at Equilibrium C2b.

$Q^*_{c,C1a}$ is the ordering quantity of the central retailer at Equilibrium C1a.

$x^{eq,Q_{nc}}_{C2b,C1a}$ is the discount triggering quantity where the expected profit for a non-central retailer at Equilibrium C2b and Equilibrium C1a is equal.

$x^{eq,Q_c}_{C1a,A}$ is the discount triggering quantity where the expected profit for the central retailer at Equilibrium C1a and Equilibrium A is equal.

Using the information in Table 13, the supplier can evaluate the expected supplier profit in each of the possible stable regions for the discount triggering quantity. Within some of the stable regions the expected supplier profit increases as the discount triggering quantity is increased. Within these regions the supplier will set the discount triggering quantity to the highest value possible while remaining in the stable region in order to receive the highest possible expected profit while keeping the retailers ordering according to the associated stable equilibrium for that region. Table 14 shows the expected supplier profit for each of the values of the discount triggering quantity that can be used in this example.

x	Q_c	Q_{nc}	Equilibrium	Supplier Profit	Central Retailer Profit	Non-Central Retailer Profit
198.459	486.255	478.789	B	7308.39	3322.13	3013.64
522.472	632.556	198.459	C1a	7685.38	2792.88	2405.77
Infinity	302.936	284.42	A	6974.21	2792.88	2236.51

Table 14: Expected supplier profit for the each of the possible supplier choices of the discount triggering quantity; c_n=18, c_r=13.

Thus in this example, where the combination of $c_n = 18$ and $c_r = 13$ is used, the supplier should set the discount triggering quantity, x, to 522.472 in order to maximize the expected supplier profit.

In order to determine the values of c_n and c_r that result in the highest expected supplier profit, the process of analysis demonstrated by the previous example must be conducted for various values of c_n and c_r. As a result trends in the expected supplier profit as well as in the expected retailer profits can been seen. Figure 44 shows the change in expected profit for the supplier and retailers as c_n changes.

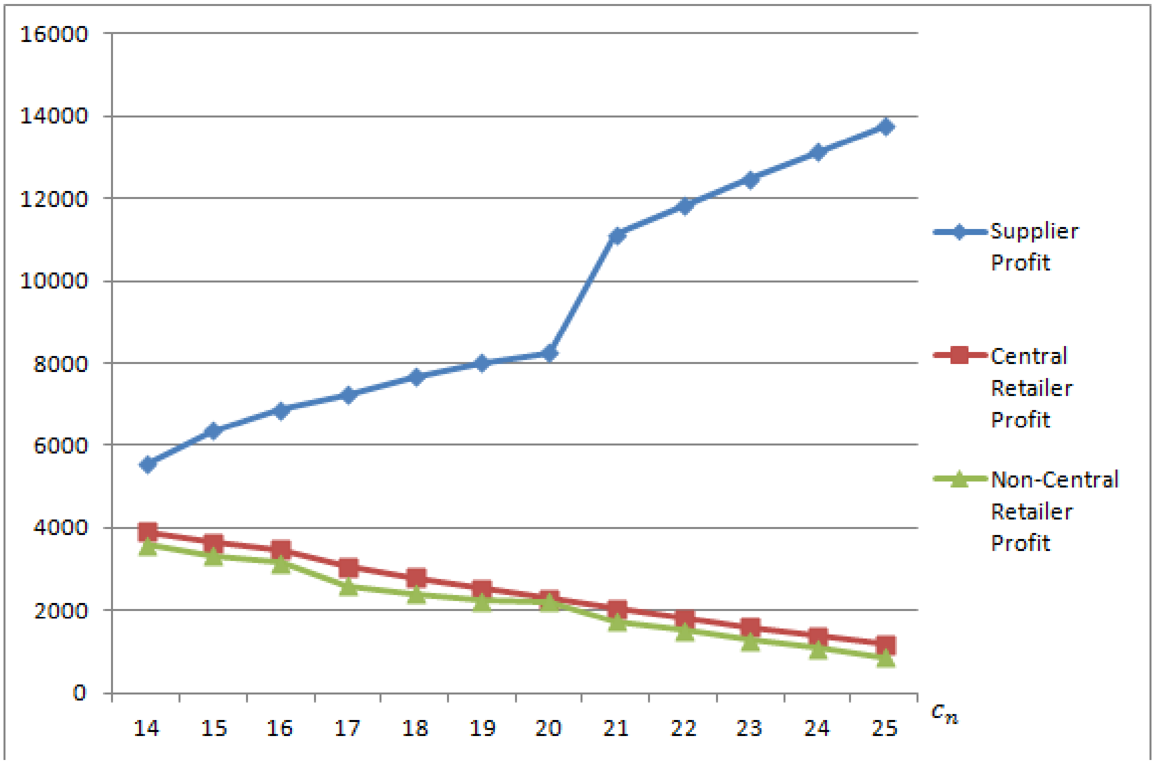

Figure 44: Changes in supplier expected profit, central retailer expected profit, and non-central retailer expected profit as c_n changes; c_r=13

Note that the expected supplier profit increases and both the expected central retailer profit and the expected non-central retailer profit both decrease as c_n increases. This result is consistent with the two retailer case that is analyzed in Chapter 3. Figure 45 shows the change in expected profit for the supplier and retailers as c_r changes.

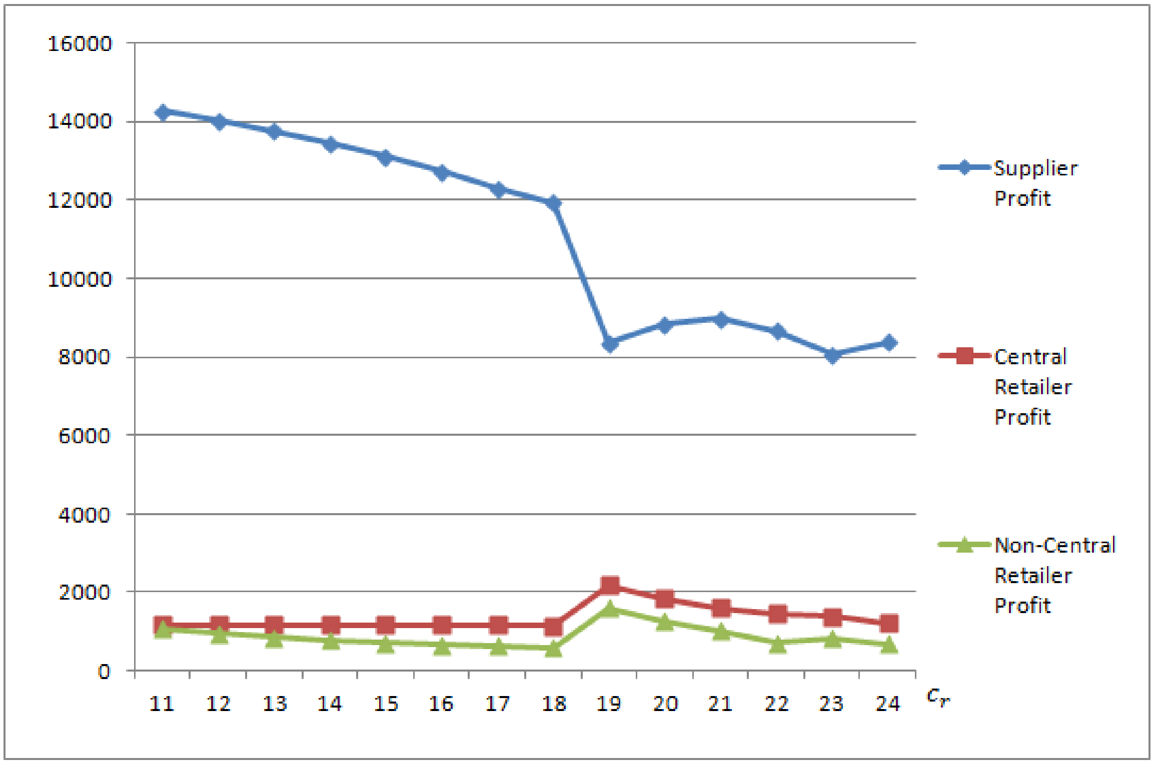

Figure 45: Changes in supplier expected profit, central retailer expected profit, and non-central retailer expected profit as c_r changes; c_n=25

Note that the highest expected supplier profit occurs when the value of c_r is the lowest. The results presented in Figure 44 and Figure 45 lead to the conclusion that the supplier will receive the highest expected profit when the supplier sets the value of c_n as high as possible within the limits of the system and c_r as low as possible within the limits of the system. This conclusion for the three retailer system is consistent with the conclusion for the two retailer system.

The abrupt changes in the slopes of the expected profit shown in Figure 44 and Figure 45 are a result of the characteristics of the stable regions of the discount triggering quantity and their impact on the largest expected supplier profit from the available stable equilibria. Consequently, the equilibria that the retailers ordering according to change as

c_n and c_r change. Table 15 and Table 16 show how expected profit changes as the equilibriums are changed in more detail.

c_n	x	Supplier Profit	Q_c	Q_{nc}	Equilibrium	Central Retailer Profit	Non-Central Retailer Profit
14	405.12	5546.86	486.255	478.789	B	3909.31	3600.82
15	340.312	6373.37	486.255	478.789	B	3633.81	3325.31
16	282.603	6874.93	486.255	478.789	B	3466.62	3158.13
17	508.022	7249.57	597.407	244.661	C1a	3061.58	2613.76
18	522.472	7685.38	23.404	583.409	C2b	2792.88	2405.77
19	549.229	8021.43	675.741	149.935	C1a	2535.46	2253.32
20	716.953	8263.42	802.184	41.9095	C1b	2287.9	2225.27
21	283.155	11127.2	486.255	478.789	B	2049.19	1740.7
22	277.313	11819	486.255	478.789	B	1818.61	1510.12
23	271.876	12487.8	486.255	478.789	B	1595.67	1287.18
24	266.765	13134.8	486.255	478.789	B	1380.01	1071.52
25	261.917	13760.5	486.255	478.789	B	1171.42	862.93

Table 15: Changes in supplier expected profit, central retailer expected profit, and non-central retailer expected profit as c_n changes; c_r=13.

c_r	x	Supplier Profit	Q_c	Q_{nc}	Equilibrium	Central Retailer Profit	Non-Central Retailer Profit
11	288.152	14251.7	718.56	715.378	B	1171.42	1062.28
12	272.541	14025.1	569.593	564.207	B	1171.42	955.09
13	261.917	13760.5	486.255	478.789	B	1171.42	862.93
14	253.78	13452.8	429.563	419.973	B	1171.42	786.65
15	247.102	13103.5	387.219	375.437	B	1171.42	725.15
16	241.364	12716	353.744	339.722	B	1171.42	676.96
17	236.27	12293.3	326.225	309.948	B	1171.42	640.68
18	226.067	11935.9	302.936	284.42	B	1138.98	582.62
19	60.86	8357.53	282.767	262.064	B	2170.3	1597.85
20	88.816	8825.05	264.968	242.156	B	1843.83	1263.85
21	108.753	8976.32	249.007	224.192	B	1614.18	1034.36
22	267.611	8667.96	23.273	279.716	C2b	1461.1	722.4
23	268.393	8056.05	0.283	268.393	C2b	1376.6	831.65
24	152.947	8383.68	208.669	178.695	B	1227.07	686.18

Table 16: Changes in supplier expected profit, central retailer expected profit, and non-central retailer expected profit as c_r changes; c_n =25.

Note that the abrupt changes in the slope of the expected profits in Figure 44 and Figure 45 are a result of changing existences of the equilibria and do not necessarily reflect changes in the choice of equilibrium as shown in Table 15 and Table 16. In such cases where an abrupt change in the slope of the expected profits occurs, the boundaries of the stable region on the retailers' chosen equilibrium are changed by the existence of other equilibria. Though these changes in the potential equilibria may not affect the retailers' decision, it allows the supplier to change the value of the discount triggering quantity while the retailers remain on the same equilibrium.

4.3.3: Comparison of the Two Retailer and Three Retailer Systems and Retailer Decisions

One of the fundamental reasons for the study of the three retailer transshipment with quantity discounts system is to evaluate the value added to each of the entities within the system through the addition of an additional retailer entering into a transshipment agreement with one of the retailers that already exists in the system. It is assumed that the decision of adding an additional retailer into the transshipment system lies with the retailers and with the supplier. This is because the transshipment agreement between the retailers is not controlled by the supplier. The supplier only controls the form and cost parameters of the contract for purchase of their product by the retailers. Thus, knowing the value of an additional transshipment partner is important to the retailers to determine if it is advantageous for them to add an additional retailer into the system. The suppliers understanding of this value helps the supplier determine if the addition of another retailer into the transshipment system should be encouraged or discouraged. Table 17 gives a comparison between the two retailer transshipment system and the three retailer transshipment system. It is assumed that the third retailer in the two retailer transshipment system acts as an independent newsvendor, unable to transship, and orders according to Equation 3.

Two Retailer Transshipment with Quantity Discount System							
x	Q_i	Q_j	Q_{news}	Retailer i Profit	Retailer j Profit	Newsvendor Profit	Supplier Profit
337.231	899.522	27.19	169.46	2596.12	2596.12	458.106	10456.3
Three Retailer Transshipment with Quantity Discount System							
x	Q_c	$Q_{nc,i}$	$Q_{nc,j}$	**Central Retailer Profit**	**Non-Central Retailer i Profit**	**Non-Central Retailer j Profit**	**Supplier Profit**
288.152	715.378	718.56	715.378	1171.42	1062.28	1062.28	14251.7

Table 17: Comparison between two retailer and three retailer transshipment models when using c_n=25, c_r=11, and exponential demand with mean of 200

From Table 17 it can be seen that the two retailers in the two retailer transshipment system have an expected profit that is more than double the expected profit of for any of the retailers in the three retailer system. Thus, it will result in a reduced expected profit for both retailers if either retailer from the two retailer system decides to acquire an additional transshipment partner. However, if a second transshipment partner is added by either of the two retailers from the two retailer transshipment system, then the supplier's expected profit will increase. Additionally, in the three retailer transshipment case, the newsvendor's expected profit more than doubles by entering into the transshipment agreement.

Chapter 5: Analysis of Different Quantity Discounts in a Two Retailer Transshipment System

Previously, a two-block tariff quantity discount has been used in the analysis of the two retailer transshipment with quantity discounts (Chapter 3) case as well as in the three retailer transshipment with a central retailer and quantity discounts case (Chapter 4). In this chapter, we examine three types of quantity discounts defined in (Dolan, 1987), which are the two-block tariff, two-part tariff, and the all-units quantity discounts in a two retailer transshipment system. Analyzing the different quantity discount types in a transshipment scenario is important to the supplier in order for the supplier to determine which quantity discount best achieves the supplier's goals when combined with the retailer's ability to transship.

5.1: Two-Block Tariff Quantity Discount

A two-block tariff quantity discount is defined (Dolan, 1987), as:

$$R(q) = \begin{cases} p_1 q, & 0 \le q \le x \\ p_1 x + p_2(q - x) & \text{for } q > x. \end{cases}$$

Where $R(q)$ is the total charged to the retailer for quantity q. Here the retailers pay a price p_1 that is charged per unit for all units up to a quantity x and all units purchased greater than x are charged a price p_2 per unit (where $p_1 > p_2$). In the two retailer transshipment with quantity discounts scenario, we define the expected profit for retailer i when a two-block tariff quantity discount is offered, π_i^{TBT}, as:

$$\pi_i^{TBT} = \begin{cases} \pi_i^{TBT,d}, & Q_i \le x \\ \pi_i^{TBT,r}, & Q_i > x \end{cases} \qquad (9)$$

For the purposes of this analysis define the expected profit for retailer i while ordering according to the function distinguished by the normal cost from the supplier, $\pi_i^{TBT,d}$, as :

$$\pi_i^{TBT,d}(Q_i, Q_j) = h(Q_i, Q_j) - c_n Q_i \qquad (47)$$

and define the expected profit for retailer i while ordering according to the function distinguished by the discounted cost from the supplier, $\pi_i^{TBT,r}$, as:

$$\pi_i^{TBT,r}(Q_i, Q_j) = h(Q_i, Q_j) - x(c_n - c_r) - c_r Q_i \qquad (48)$$

where

$$h(Q_i, Q_j) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} \qquad (49)$$

Carrying out the expectation operator from Equation 49 and rewriting it in integral form results in the following equation:

$$h(Q_i,Q_j) = r_i\left[\int_0^{\infty}\int_0^{Q_i} D_i f(D_i)f(D_j)dD_i dD_j + \int_0^{\infty}\int_{Q_i}^{\infty} Q_i f(D_i)f(D_j)dD_i dD_j\right.$$
$$+ \int_0^{Q_j}\int_{Q_i}^{Q_i+Q_j-D_j}(D_i - Q_i)f(D_i)f(D_j)dD_i dD_j$$
$$\left.+ \int_0^{Q_j}\int_{Q_i+Q_j-D_j}^{\infty}(Q_j - D_j)f(D_i)f(D_j)dD_i dD_j\right]$$
$$- c_{ji}\left[\int_0^{Q_j}\int_{Q_i}^{Q_i+Q_j-D_j}(D_i - Q_i)f(D_i)f(D_j)dD_i dD_j\right.$$
$$\left.+ \int_0^{Q_j}\int_{Q_i+Q_j-D_j}^{\infty}(Q_j - D_j)f(D_i)f(D_j)dD_i dD_j\right]$$
$$+ (c_{ij} - \tau_{ij})\left[\int_0^{Q_i}\int_{Q_i+Q_j-D_i}^{\infty}(Q_i - D_i)f(D_i)f(D_j)dD_i dD_j\right.$$
$$\left.+ \int_0^{Q_i}\int_{Q_j}^{Q_i+Q_j-D_j}(D_j - Q_j)f(D_i)f(D_j)dD_i dD_j\right]$$
$$+ s_i\left[\int_{Q_j}^{Q_i+Q_j}\int_0^{Q_i+Q_j-D_j}(Q_i - D_i + Q_j - D_j)\, f(D_i)f(D_j)dD_i dD_j\right.$$
$$\left.+ \int_0^{Q_j}\int_0^{Q_i}(Q_i - D_i)f(D_i)f(D_j)dD_i dD_j\right]$$
$$- p_i\left[\int_0^{Q_j}\int_{Q_i+Q_j-D_j}^{\infty}(D_i + D_j - Q_i - Q_j)\, f(D_i)f(D_j)dD_i dD_j\right.$$
$$\left.+ \int_{Q_j}^{\infty}\int_{Q_i}^{\infty}(D_i - Q_i)f(D_i)f(D_j)dD_i dD_j\right] \tag{50}$$

The response functions in this case are found by taking the derivative of Equation 47 and Equation 48 with respect to Q_i. Similar to Equation 15, the response function for retailer

i while ordering according to the function distinguished by the normal cost from the supplier is given in Equation 51.

$$\frac{\partial \pi_i^{TBT,d}}{\partial Q_i} = g_i(Q_i, Q_j) - c_n \tag{51}$$

Similar to Equation 14, the response function for retailer i while ordering according to the function distinguished by the discounted cost from the supplier is given in Equation 52

$$\frac{\partial \pi_i^{TBT,r}}{\partial Q_i} = g_i(Q_i, Q_j) - c_r \tag{52}$$

where $g_i(Q_i, Q_j)$ is the derivative of $h(Q_i, Q_j)$ with respect to Q_i and is defined in Chapter 3 as:

$$\begin{aligned} g_i(Q_i, Q_j) = r_i &\left[\int_0^{\infty}\int_{Q_i}^{\infty} f(D_i)f(D_j)dD_i dD_j - \int_0^{Q_j}\int_{Q_i}^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j\right] \\ &+ c_{ji}\left[\int_0^{Q_j}\int_{Q_i}^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j\right] \\ &+ (c_{ij} - \tau_{ij})\left[\int_0^{Q_i}\int_{Q_i+Q_j-D_i}^{\infty} f(D_i)f(D_j)dD_i dD_j\right] \\ &+ s_i\left[\int_{Q_j}^{Q_i+Q_j}\int_0^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j \right. \\ &\left. + \int_0^{Q_j}\int_0^{Q_i} f(D_i)f(D_j)dD_i dD_j\right] \\ &+ p_i\left[\int_0^{Q_j}\int_{Q_i+Q_j-D_j}^{\infty} f(D_i)f(D_j)dD_i dD_j - \int_{Q_j}^{\infty}\int_{Q_i}^{\infty} f(D_i)f(D_j)dD_i dD_j\right] \end{aligned} \tag{13}$$

5.2: A Modified Two-Part Tariff Quantity Discount

A two-part tariff quantity discount is defined as:

$$R(q) = \begin{cases} F + pq, & q > 0, \\ 0, & q = 0, \end{cases} \text{ where } F > 0.$$

Here the retailers pay a fixed cost *F*, in order to purchase units at a price *p*, that is charged per unit.

For the purposes of comparing the different quantity discounts, a variation of the two-part tariff quantity discount, as defined in (Dolan, 1987) is used. Consider that when the two-part tariff quantity discount is offered, the retailer has the choice of paying the fixed cost to be able to purchase all units at a discounted price p_2 per unit, or not paying the fixed cost and paying price p_1 per unit (where $p_1 > p_2$).

$$R(q) = \begin{cases} F + p_2 q, \\ p_1 q, \end{cases} \quad \text{where } F > 0.$$

In the two retailer transshipment with quantity discounts scenario, we define the expected profit for retailer *i* when a modified two-part tariff quantity discount is offered, π_i^{TPT}, as:

$$\pi_i^{TPT} = \begin{cases} \pi_i^{TPT,d} \\ \pi_i^{TPT,r} \end{cases} \tag{53}$$

For the purposes of this analysis we define the expected profit for retailer *i* while ordering according to the function distinguished by the normal cost from the supplier, $\pi_i^{TPT,d}$, as :

$$\pi_i^{TPT,d}(Q_i, Q_j) = h(Q_i, Q_j) - c_n Q_i \tag{54}$$

and we define the expected profit for retailer *i* while ordering according to the function distinguished by the discounted cost from the supplier, $\pi_i^{TPT,r}$, as:

$$\pi_i^{TPT,r}(Q_i, Q_j) = h(Q_i, Q_j) - F - c_r Q_i \tag{55}$$

Similar to the case described in the previous section, the response function for the modified two-part tariff case is found by taking the derivative of the profit function, Equation 53, with respect to Q_i. Thus, the response function for retailer *i* while ordering according to the function distinguished by the normal cost from the supplier is given in Equation 56.

$$\frac{\partial \pi_i^{TPT,d}}{\partial Q_i} = g_i(Q_i, Q_j) - c_n \tag{56}$$

The response function for retailer *i* while ordering according to the function distinguished by the discounted cost from the supplier is given in Equation 57.

$$\frac{\partial \pi_i^{TPT,r}}{\partial Q_i} = g_i(Q_i, Q_j) - c_r \tag{57}$$

5.3: All-Units Quantity Discount

Recall, in Chapter 2, that an all-units quantity discount is defined as:

$$R(q) = \begin{cases} p_1 q & \text{if} \quad 0 \le q \le x, \\ p_2 q & \text{if} \quad \quad q \ge x, \end{cases} \qquad \text{where } p_1 > p_2.$$

Here a retailer pays a price p_1 per unit purchased if the ordering quantity is below the discount triggering quantity, *x*, and pay a price p_2 per unit purchased if the ordering quantity is above the quantity discount point (where $p_1 > p_2$).

In the two retailer transshipment with quantity discounts scenario, we define the expected profit for retailer *i* when an all-units quantity discount is offered, π_i^{AU}, as:

$$\pi_i^{AU} = \begin{cases} \pi_i^{AU,d}, & Q_i \le x \\ \pi_i^{AU,r}, & Q_i > x \end{cases} \tag{58}$$

Define the expected profit for retailer *i* while ordering according to the function distinguished by the normal cost from the supplier, $\pi_i^{AU,d}$, as :

$$\pi_i^{AU,d}(Q_i, Q_j) = h(Q_i, Q_j) - c_n Q_i \tag{59}$$

and define the expected profit for retailer *i* while ordering according to the function distinguished by the discounted cost from the supplier, $\pi_i^{AU,r}$, as:

$$\pi_i^{AU,r}(Q_i, Q_j) = h(Q_i, Q_j) - c_r Q_i \tag{60}$$

Similar to the cases described in the two previous sections, the response function for the all-units quantity discount case is found by taking the derivative of the profit function, Equation 58, with respect to Q_i. Thus, the response function for retailer *i* while ordering according to the function distinguished by the normal cost from the supplier is given in Equation 61.

$$\frac{\partial \pi_i^{AU,d}}{\partial Q_i} = g_i(Q_i, Q_j) - c_n \tag{61}$$

The response function for retailer *i* while ordering according to the function distinguished by the discounted cost from the supplier is given in Equation 62.

$$\frac{\partial \pi_i^{AU,r}}{\partial Q_i} = g_i(Q_i, Q_j) - c_r \tag{62}$$

5.4: Comparison of Different Quantity Discounts

This section gives an analysis of the three types of quantity discounts (two-block tariff, modified two-part tariff, and all-units) discussed in this chapter. First, an analysis of the response functions of the different quantity discounts is presented. The resulting supplier and retailer decisions are then analyzed for the three types of quantity discounts. Lastly, the resulting profits are compared and conclusions are made about the impact that the different quantity discounts have on supplier and retailer profits.

A comparison of the mathematical derivative expressions used in the response functions representing the response of retailer *i* while on the response curve distinguished by the normal cost from the supplier shows these components of the response functions are equivalent, such that:

$$\frac{\partial \pi_i^{TBT,d}}{\partial Q_i} = \frac{\partial \pi_i^{TPT,d}}{\partial Q_i} = \frac{\partial \pi_i^{AU,d}}{\partial Q_i} = g_i(Q_i, Q_j) - c_n \tag{63}$$

Similarly, a comparison of the mathematical derivative expressions used in the response functions representing the response of retailer *i* while on the response curve distinguished by the discounted cost from the supplier shows that these components of the response functions are equivalent, such that:

$$\frac{\partial \pi_i^{TBT,r}}{\partial Q_i} = \frac{\partial \pi_i^{TPT,r}}{\partial Q_i} = \frac{\partial \pi_i^{AU,r}}{\partial Q_i} = g_i(Q_i, Q_j) - c_r \qquad (64)$$

Though the components that make up the response functions do not differ between different quantity discount types, the response functions in their entirety do differ between different quantity discount types. This is due to the different characteristics of the quantity discounts that determine when and by what mechanism the discount engages. The components must be composed into the entire response function for each case based on the logic of the discount being considered. This determines which response curve the retailer's response is based on. Likewise, the existence of different potential equilibrium points is based on the interaction between the two retailer's overall response functions for each type of quantity discount.

In the two-block tariff case, the discount triggering quantity, *x*, determines the point at which the retailer's response jumps between the normal and discounted response curves. Additionally, the relative value of *x* determines the existence of the four possible equilibria. This can be seen in Figure 46. Note the discontinuity in the response function at the value of *x* for each retailer.

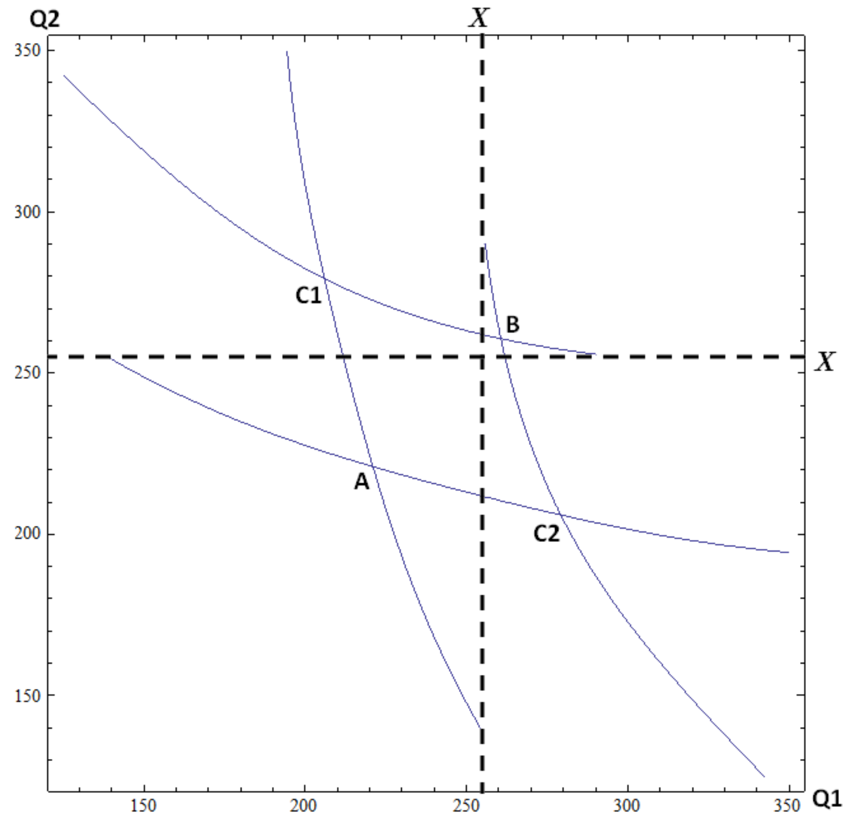

Figure 46: Two-block tariff quantity discount response function. The response functions are described in detail in…

The two-part tariff case differs from the other cases in that there is no discontinuity in the response function. Thus, the existence of the equilibria is not determined by any variable other than the cost parameters. The four equilibria are always in existence. An example of the response function when a two-part tariff quantity discount is used is shown in Figure 47.

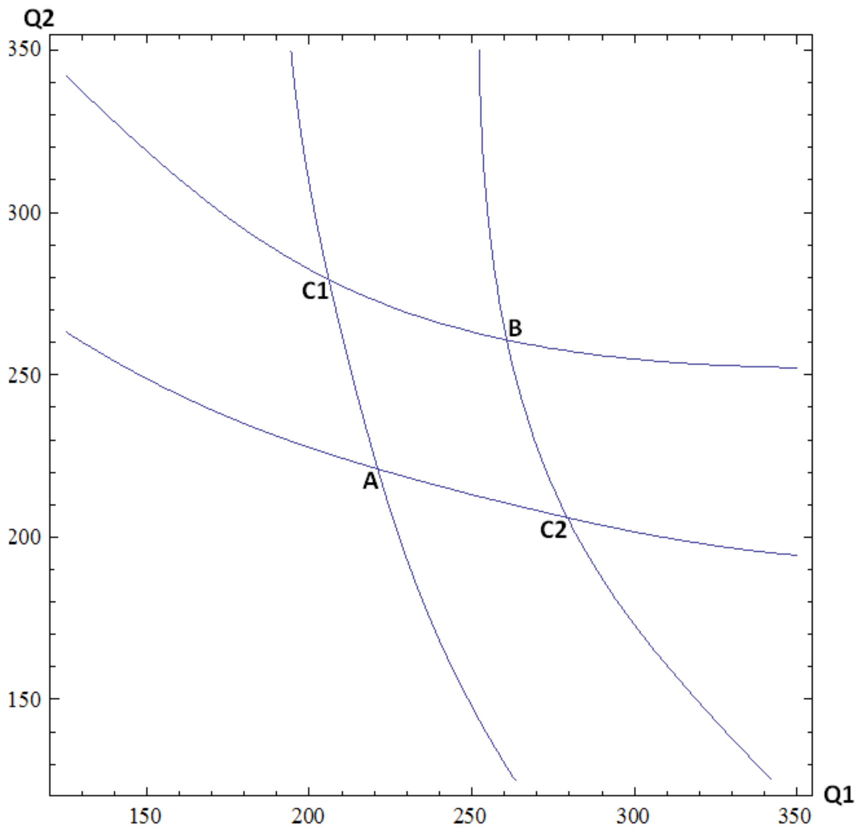

Figure 47: Two-part tariff quantity discount response function

The response function of the all-units quantity discount case is similar to the two-block tariff case due to both of the response functions having a discontinuity at the point of the discount triggering quantity. Here there is a discontinuity at the discount triggering quantity that determines on which response curve the retailer's response is based. Also, like the two-block tariff case, in the all-units case, *x* determines the existence of the four equilibria. A response function graph of the all-units quantity discount case can be seen in Figure 48.

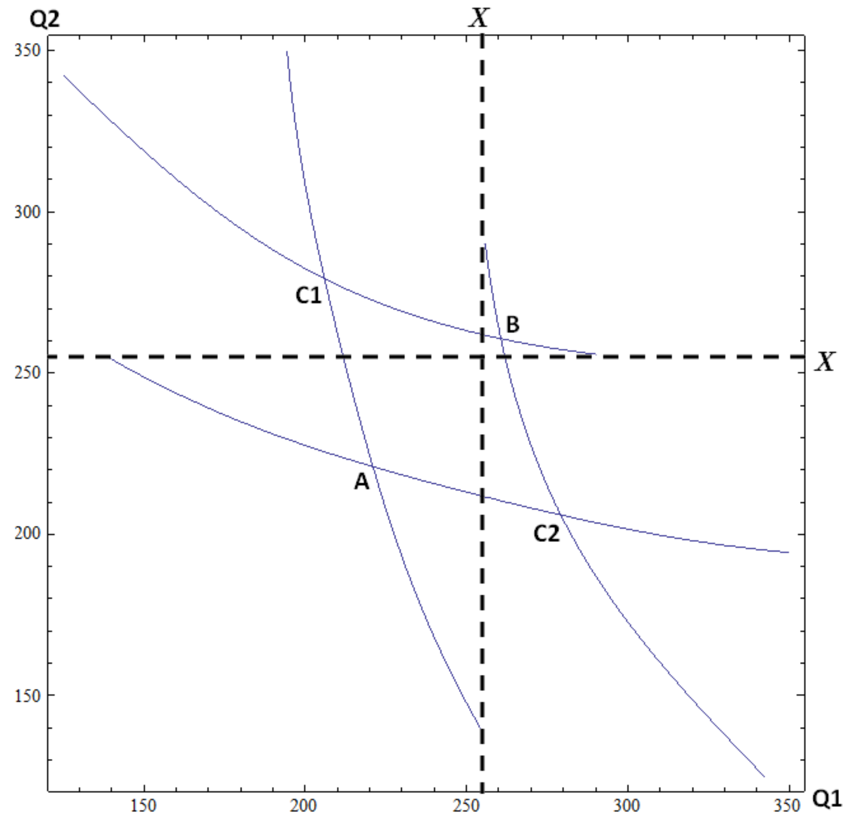

Figure 48: All units quantity discount response function

At each of the potential equilibria that are a result of the response function interactions between the retailers, the possible expected profits can be determined for each quantity discount type. Figure 49 shows the expected profit for retailer *i* for each of the equilibria when a two-block tariff quantity discount is used. Note that the equilibria only exist within certain ranges, which are controlled by the value of *x*.

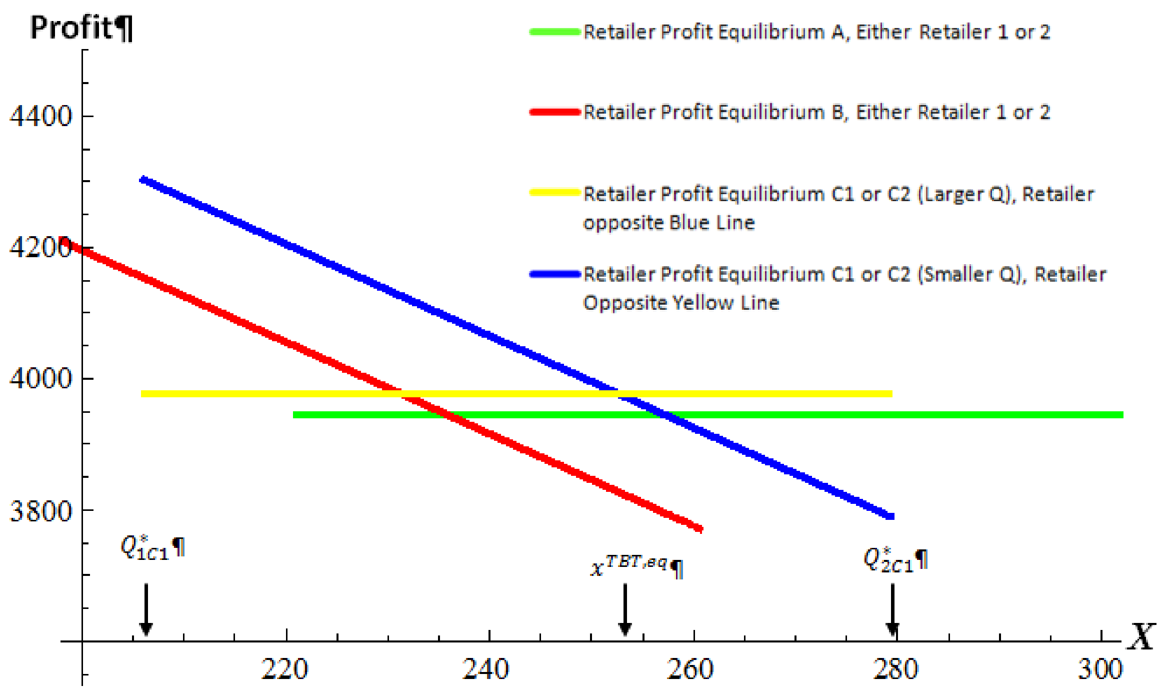

Figure 49: Retailer's expected profit as x changes when a two-block tariff quantity discount is offered.

The two-block tariff case offers the largest number of possible stable equilibria for the supplier to choose from, when deciding the value of the discount triggering quantity, of the three different quantity discount cases. Table 18 gives an overview of the retailer's choice within each of those stable ranges and the supplier's profit trend within those ranges.

<u>Retailer's Choice</u>		
x	**Choice**	**Supplier's Profit Trend**
0 to Q^*_{1C1}	B	Increasing as x increases
Q^*_{1C1} to $x^{TBT,eq}$	Unstable	Unstable
$x^{TBT,eq}$	C	-
$x^{TBT,eq}$ to Q^*_{2C1}	Unstable	Unstable
Q^*_{2C1} to ∞	A	Constant as x increases

Table 18: Retailers's choice of equilibrium given *x* when a two-block tariff quantity discount is offered.

Let $x^{TBT,eq}$ be defined as the value of *x* at which the expected profits for retailer *i* and retailer *j* are equal while the retailers are ordering according to a C equilibrium and a two-block tariff quantity discount is offered. Therefore

$$x^{TBT,eq} = \frac{m(Q_i,Q_j)}{c_n - c_r} \tag{65}$$

where

$$\begin{aligned} m(Q_i,Q_j) = {} & (Q_i - Q_j) \\ & * \Bigg[(c - r - p)\int_0^{Q_i}\int_{Q_i+Q_j-D_i}^{\infty} f(D_i)f(D_j)dD_i dD_j + (c - \tau \\ & - s)\int_0^{Q_i}\int_{Q_j}^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j\Bigg] \\ & + \left(-\tau * Q_i + (c - s) * (Q_i + Q_j)\right)\int_{Q_i}^{Q_j}\int_0^{Q_i+Q_j-D_j} f(D_i)f(D_j)dD_i dD_j \\ & + Q_j(\tau - 2c + 2r + 2p)\int_{Q_i}^{Q_j}\int_{Q_i+Q_j-D_i}^{\infty} f(D_i)f(D_j)dD_i dD_j \\ & + (2c - 2r - p - \tau)\int_{Q_i}^{Q_j}\int_{Q_i+Q_j-D_i}^{\infty} D_j f(D_i)f(D_j)dD_i dD_j \\ & + (-2c + \tau + s)\int_{Q_i}^{Q_j}\int_0^{Q_i+Q_j-D_j} D_j f(D_i)f(D_j)dD_i dD_j \\ & + p\int_{Q_j}^{\infty}\int_{Q_i}^{\infty}(D_j - D_i)f(D_i)f(D_j)dD_i dD_j \\ & - p\int_{Q_i}^{Q_j}\int_{Q_i+Q_j-D_i}^{\infty} D_i f(D_i)f(D_j)dD_i dD_j \\ & + s\int_0^{Q_j}\int_0^{Q_i}(D_j - D_i)f(D_i)f(D_j)dD_i dD_j \\ & + s\int_{Q_i}^{Q_j}\int_0^{Q_i+Q_j-D_j} D_i f(D_i)f(D_j)dD_i dD_j + Q_j(c_r - c_n) \end{aligned} \tag{66}$$

The two-part tariff quantity discount case results in a similar retailer expected profit graph as in the two-block tariff quantity discount case. Figure 50 shows the expected retailer profit when a two-part tariff quantity discount is offered. Note that the difference between Figure 49 and Figure 50 is that when a two-part tariff quantity discount is offered, the existences of the equilibria are not controlled by any variable that is characteristic to the quantity discount where as in the two-block tariff quantity discount case, the existence of the equilibria is controlled by *x*. The result is that in the two-part tariff quantity discount case, there is only one stable point for *F*.

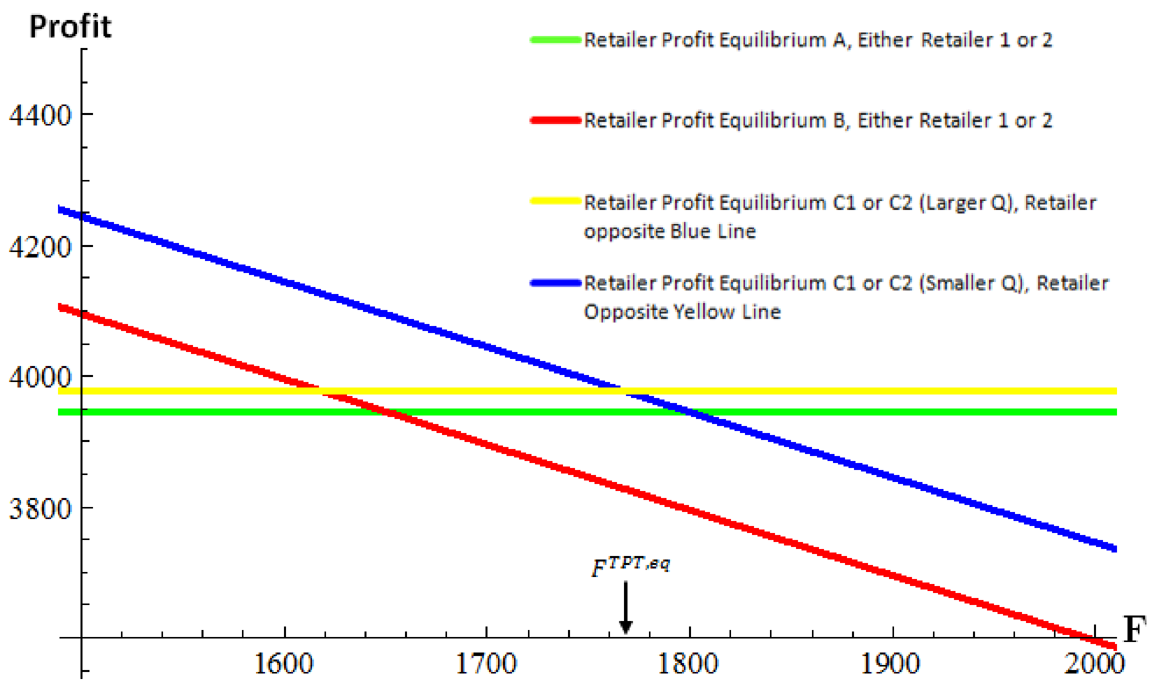

Figure 50: Retailer's expected profit as *F* changes when a two-part tariff quantity discount is offered.

Table 19 gives an overview of the retailer's choice within each of stable region when a two-part tariff quantity discount is used.

Retailer's Choice		
F	**Choice**	**Supplier's Profit Trend**
0 to $F^{TPT,eq}$	Unstable	Unstable
$F^{TPT,eq}$	C	-
$F^{TPT,eq}$ to ∞	Unstable	Unstable

Table 19: Retailers's choice of equilibrium given *F* when a two-block tariff quantity discount is offered.

Note that in the two-part tariff case exactly one stable value of F exists. This is due to there being no variable controlling the existence of the equilibria. It turns out that because all potential equilibria are present at all times, only one fulfills the stability criteria. Therefore, the single existing value of F is identified by $F^{TPT,eq}$, and is defined as:

$$F^{TPT,eq} = m(Q_i, Q_j) \quad (67)$$

Though the response functions of the two-block tariff quantity discount and the all-units quantity discount cases are identical, the retailer's expected profits are not because of the differences in the discount mechanisms. Figure 51 shows the retailer's expected profit for the all-units quantity discount case.

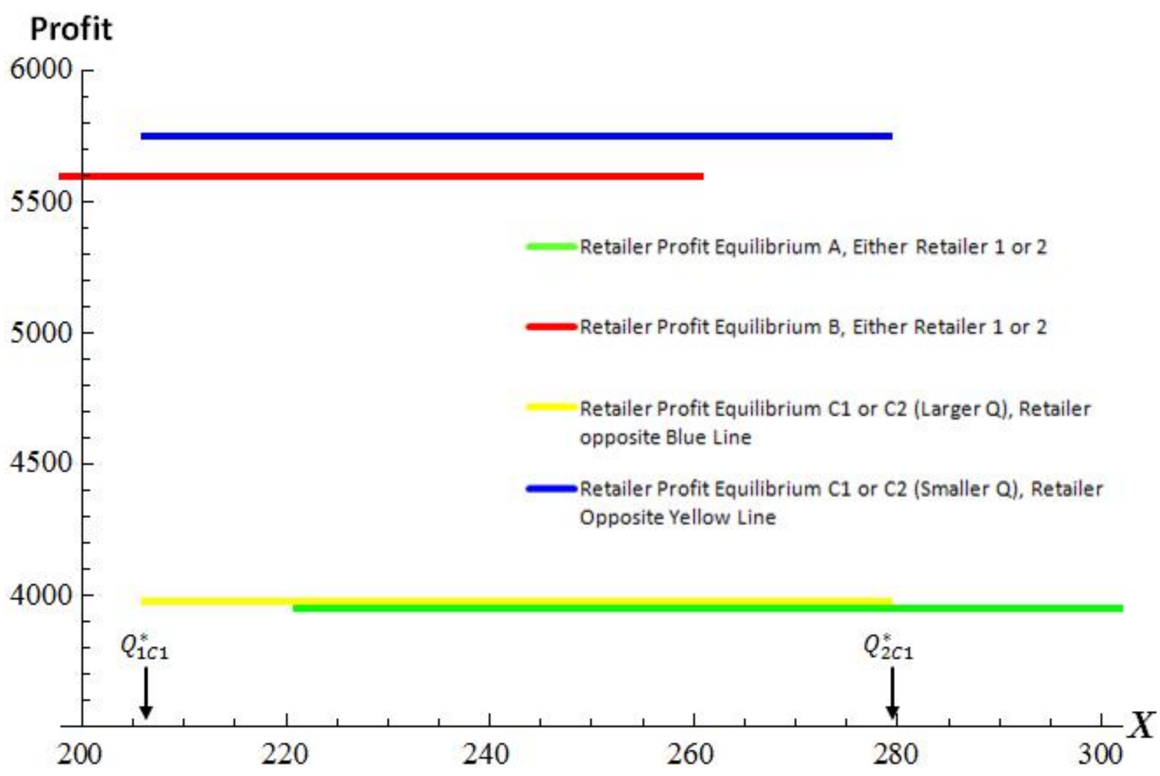

Figure 51: Retailer's expected profit as *x* changes when an all-units quantity discount is offered.

From Figure 51 it can be seen that no equilibrium meets the stability criteria when Equilibrium C is in existence. The regions of *x* where an equilibrium meeting the stability criteria exists are for smaller values of *x* and larger values of *x*. Table 20 shows the stable regions for the all-units quantity discount case.

Retailer's Choice		
x	**Choice**	**Supplier's Profit Trend**
0 to Q^*_{1C1}	B	Increasing as x increases
Q^*_{1C1} to Q^*_{2C1}	Unstable	Unstable
Q^*_{2C1} to ∞	A	Constant as x increases

Table 20: Retailers's choice of equilibrium given *x* when an all-units quantity discount is offered.

5.5: The Effects of Different Quantity Discounts on Supplier Decisions

The examination of different quantity discounts on supplier profit is an important topic because the supplier determines which quantity discount is offered to the retailers. Differences between the different quantity discounts, as related to the supplier, include the available stable regions of *x* (or *F*, in the case of the two-block tariff quantity discount) and the expected supplier profit. Thus, it is important to the supplier to analyze the effects of the different quantity discounts.

In order to determine which quantity discount type a supplier should offer, the differences in profit between the use of the different quantity discounts are examined. Table 21 shows the expected profits and possible choices for the value of the discount triggering quantity for the two-block tariff quantity discount case.

	Two-Block Tariff Quantity Discount		
Equilibrium	***x***	**Retailer Profit**	**Supplier Profit**
A	Q^*_{1C1}	$h(Q_i, Q_j) - c_n Q_i$	$c_n(Q_i + Q_j) - c_s(Q_i + Q_j)$
B	Q^*_{2C1}	$h(Q_i, Q_j) - x(c_n - c_r) - c_r Q_i$	$2x(c_n - c_r) + c_r(Q_i + Q_j) - c_s(Q_i + Q_j)$
C (high)	$x^{TBT,eq}$	$h(Q_i, Q_j) - x(c_n - c_r) - c_r Q_i$	$x(c_n - c_r) + c_r Q_i + c_n Q_j - c_s(Q_i + Q_j)$
C (low)	$x^{TBT,eq}$	$h(Q_i, Q_j) - c_n Q_i$	$x(c_n - c_r) + c_r Q_i + c_n Q_j - c_s(Q_i + Q_j)$

Table 21: Potential stable values of *x* and the associated ordering quantities, expected retailer profits, and expected supplier profits for the two-block tariff quantity discount case.

Note that in the two-block tariff quantity discount case stable values of *x* exist for all three equilibria. Section 3.3.4 describes the conditions for stability.

Table 22 shows the expected profits and possible choices for the value of the discount triggering quantity for the two-part tariff quantity discount case.

	Two-Part Tariff Quantity Discount		
Equilibrium	***F***	**Retailer Profit**	**Supplier Profit**
A	-	-	-
B	-	-	-
C (high)	$F^{TPT,eq}$	$h(Q_i, Q_j) - F - c_r Q_i$	$F + c_r Q_i + c_n Q_j - c_s(Q_i + Q_j)$
C (low)	$F^{TPT,eq}$	$h(Q_i, Q_j) - c_n Q_i$	$F + c_r Q_i + c_n Q_j - c_s(Q_i + Q_j)$

Table 22: Potential stable values of *F* and the associated ordering quantities, expected retailer profits, and expected supplier profits for the two-part tariff quantity discount case.

Here the definition of stability adjusted to account for *F*, such that a stable region is defined as a region, distinguished by the fixed cost, where

1. the expected profit for retailer *i* is equal to the expected profit for retailer *j* and

2. the two retailers are at an equilibrium that has a larger expected profit for both retailers than any other equilibrium that is in existence for a particular value of *F*.

Note that in the two-part tariff quantity discount case, only the C equilibrium is obtainable as a stable solution for the retailer's ordering quantities.

Table 23 shows the expected profits and possible choices for the value of the discount triggering quantity for the all-units quantity discount case.

	All-Units Quantity Discount		
Equilibrium	***x***	**Retailer Profit**	**Supplier Profit**
A	Q^*_{1C1}	$h(Q_i, Q_j) - c_n Q_i$	$c_n(Q_i + Q_j) - c_s(Q_i + Q_j)$
B	Q^*_{2C1}	$h(Q_i, Q_j) - c_r Q_i$	$c_r(Q_i + Q_j) - c_s(Q_i + Q_j)$
C (high)	-	-	-
C (low)	-	-	-

Table 23: Potential stable values of *x* and the associated ordering quantities, expected retailer profits, and expected supplier profits for the all-units quantity discount case.

As Table 23 shows, stable values of *x* can be found for Equilibrium A and Equilibrium B, but there is no stable value of *x*, for Equilibrium C.

Keep in mind that ordering quantities are different within each table and are labeled by the equilibrium representing each particular set of ordering quantities. Though this prevents direct comparisons to be made between expected profits at different equilibriums, direct comparisons can be made between different quantity discount types when the retailers are ordering according to a common equilibrium. This is because the ordering quantities for each equilibrium remain constant between the different quantity discount types.

A direct comparison can be made for the Equilibrium A and Equilibrium B solutions between the two-block tariff quantity discount case and the all-units quantity discount case. It is evident from the supplier profits shown in Table 21 and Table 23 that the two-block tariff quantity discount results in an expected supplier profit that is equal to that of the all-units quantity discount when the retailers order according to Equilibrium A. Additionally, the two-block tariff quantity discount results in an expected supplier profit that is greater than that of the all-units quantity discount when the retailer order according to Equilibrium B.

Proposition 3: The response function for the all-units quantity discount and the response function for the two-block tariff quantity discount result in the same equilibria

Proof: The response curve functions are found by taking the derivative of the retailer i expected profit functions with respect to Q_i. This process leads to the response curve functions given in Equation 51 and Equation 52 for the two-block tariff quantity discount case and in Equation 61 and Equation 62 for the all-units quantity discount case. Thus, it can be shown that the two response curves that are used in the response function are identical when comparing the two-block tariff and all-units quantity discounts.

In addition to the response curves being identical, the condition in the response functions that determines which response curve is being used must also be the same. This indeed holds as well, because in both cases the discount triggering quantity, x

determines which response curve is used, such that for each quantity discount case, when $0 \leq Q_i \leq x$, the response curve associated with the normal cost from supplier is used and when $Q_i \geq x$ the response curve associated with the discounted cost from the supplier is used.

Since the response functions of these two cases share these similarities, the equilibria will be the same in both cases. Additionally since the condition for determining which response curve is being used is the same for both quantity discount cases, the existence for the equilibria will always be the same as long as the same discount triggering quantity is being used in each case. ■

Proposition 4: The expected supplier profit when a two-block tariff quantity discount is used is greater than or equal to the expected supplier profit when an all-units quantity discount is used

Proof: Due to the unstable regions on *x* in the all-units quantity discount case, it is only possible to get a stable equilibrium where the retailers order according to Equilibrium A or Equilibrium B. The response functions of both quantity discount types result in the same Equilibrium points, thus direct comparisons can be made between the different quantity discount types where the retailers are ordering according to a common equilibrium. When the retailers are ordering according to Equilibrium A, the expected supplier profit for both the two-block tariff quantity discount and the all-units quantity discount are equal; such that:

$$\pi^{TBT,s}(Q_{i,A}, Q_{j,A}) = \pi^{AU,s}(Q_{i,A}, Q_{j,A}) = c_n(Q_i + Q_j) - c_s(Q_i + Q_j) \qquad (68)$$

When the retailers are ordering according to Equilibrium B, then the expected supplier profit for the case where a two-block tariff quantity discount is used is strictly greater than the expected supplier profit for the case where an all-units quantity discount is used; such that:

$$\pi^{TBT,s}\left(Q_{i,B}, Q_{j,B}\right) = \pi^{AU,s}\left(Q_{i,B}, Q_{j,B}\right) + 2x(c_n - c_r) \tag{69}$$

Thus, when comparing the expect supplier profits, the all-units quantity discount case is at best, equal to the two-block tariff quantity discount, which only occurs when $\pi^{TBT,s}\left(Q_{i,A}, Q_{j,A}\right) > \pi^{TBT,s}\left(Q_{i,B}, Q_{j,B}\right)$ and $\pi^{TBT,s}\left(Q_{i,A}, Q_{j,A}\right) > \pi^{TBT,s}\left(Q_{i,C1}, Q_{j,C1}\right)$. The expected suppler profit for the two-block tariff quantity discount case is greater than the expected profit for the two-block tariff case when either $\pi^{TBT,s}\left(Q_{i,B}, Q_{j,B}\right) > \pi^{TBT,s}\left(Q_{i,A}, Q_{j,A}\right)$ or $\pi^{TBT,s}\left(Q_{i,C1}, Q_{j,C1}\right) > \pi^{TBT,s}\left(Q_{i,A}, Q_{j,A}\right)$ ■

Thus, the two-block tariff quantity discount is the preferred choice for a quantity discount over the all-units quantity discount for the following reasons:

1. When the retailers order in accordance to Equilibrium B, the expected supplier profit when a two-block tariff quantity discount is used is always greater than the expected supplier profit when an all-units quantity discount is used.
2. When the retailers order in accordance to Equilibrium C, there is no stable value of *x* in the all-units quantity discount case, thus the all-units quantity discount is not an option if the supplier wishes to set the discount triggering quantity within this range of existence.

3 When the retailers order in accordance to Equilibrium A, the expected supplier profit is equal for both discount schemes.

A direct comparison can be made for the Equilibrium C solution between the two-block tariff quantity discount case and the two-part tariff quantity discount case. It is evident from the supplier profits shown in Table 21 and Table 22 that the two-block tariff quantity discount results in an expected supplier profit that is equal to that of the two-part tariff quantity discount when the retailers order according to Equilibrium C.

Proposition 5: When the retailers are ordering according to Equilibrium C, the expected supplier profit when a two-block tariff quantity discount is used is equal to the expected supplier profit when a two-part tariff quantity discount is used.

Proof: Due to the unstable regions on *F* in the two-part tariff quantity discount case, it is only possible to get a stable equilibrium where the retailers order according to Equilibrium C. Again, the response functions of both quantity discount types result in the same Equilibrium points, allowing direct comparisons to be made between the different quantity discount types where the retailers are ordering according to a common equilibrium. When the retailers are ordering according to Equilibrium C, the expected supplier profit for both the two-block tariff quantity discount case and the two-part tariff quantity discount case are equal. Given that $F^{TPT,eq} = m(Q_i, Q_j)$ and $x^{TBT,eq} = \frac{m(Q_i,Q_j)}{c_n - c_r}$; we can solve for $m(Q_i, Q_j)$ in both equations and then set them equal to each other resulting in:

$$F^{TPT,eq} = x^{TBT,eq}(c_n - c_r) \tag{70}$$

Thus it is evident that

$$\pi^{TBT,s}\left(Q_{i,C1}, Q_{j,C1}\right) = \pi^{TPT,s}\left(Q_{i,C1}, Q_{j,C1}\right) = x(c_n - c_r) + c_r Q_i + c_n Q_j - c_s(Q_i + Q_j) \tag{71}$$

■

Therefore, the two-block tariff quantity discount is the preferred choice for a quantity discount over the two-part tariff quantity discount for the following reasons:

1 When the retailers order in accordance to Equilibrium C, the expected supplier profit is equal for the two quantity discount types.

2 When the retailers order in accordance to Equilibrium A or Equilibrium B, there is no stable value of *F* in the two-part tariff quantity discount case, thus the two-part tariff quantity discount is not an option if the supplier wishes to set the discount triggering quantity within this range of existence.

The two-block tariff quantity discount is the preferred choice by the supplier over the two-part tariff quantity discount and the all-units quantity discount. The two-block quantity discount offers more options to the supplier when setting the discount triggering quantity than the other two quantity discount types examined. The two-part tariff quantity discount and the all-units quantity discount never result in a greater expected supplier profit than the two-block tariff quantity discount. Additionally, the two-block quantity discount has a greater supplier expected profit when the retailers order according to Equilibrium B than the all-units quantity discount.

Chapter 6: Summary and Conclusions

This dissertation extends the current literature by analyzing the impact of the supplier offering quantity discounts in transshipment systems. More specifically, this dissertation contributes to the knowledge in inventory systems by studying quantity discounts in two retailer transshipment systems, three retailer transshipment systems where one retailer is central, and by analyzing different quantity discount schemes in the two retailer transshipment system. Many insights regarding transshipment systems are gained through the research presented in this dissertation. The insights gained through this research are both important for real-world use and to future research on the topic of transshipment.

This dissertation first develops a model including a two-block tariff quantity discount pricing contract that is offered by a supplier to two independent newsvendor retailers. A methodology is developed for carefully building expected profit functions and response functions in a two retailer transshipment with quantity discounts system. These models are similar to those that exist in literature with the exception that quantity discount structure turns the functions into piecewise functions. Instead of having a single cost per unit purchased from the supplier, the quantity discount scheme creates a discontinuity at the discount triggering quantity. Using the two-block tariff quantity discount scheme in a transshipment model results in the two retailers having four potential equilibrium quantity combinations to consider, rather than just one when the supplier does not offer a pricing contract with a quantity discount. Two of the potential retailer equilibrium combinations have symmetric retailer ordering quantities, while the other two potential retailer equilibrium combinations have non-symmetric ordering quantities. This is in spite of otherwise symmetric cost and demand parameters.

In order to analyze this system, a methodology for analyzing multiple potential equilibria is developed. This methodology includes extending the understanding of equilibrium behavior in these systems by defining stable values of the discount triggering quantity. A stable value of the discount triggering quantity is defined as a value of the discount triggering quantity where the expected profit for all retailers is at an equilibrium that has a larger expected profit for all retailers than any other equilibrium that is in existence for a particular value of the discount triggering quantity. The resolving of supplier and retailer choices is done based upon this new stability criteria. Using this definition we find that there exists a single value of the discount triggering quantity that results in the two symmetric retailers having different ordering quantities. Additionally, the expected profit for both retailers and the supplier is greater than that of the transshipment model without quantity discounts.

The quantity discount cost parameters are chosen by the supplier. Strategically, the supplier should choose the pricing parameters within a range so that the retailers have a non-symmetric equilibrium to consider, since this can lead to the highest profits for the supplier. Also, the supplier must carefully choose a stable value of the discount triggering quantity so that the retailers have equal profits, allowing the equilibrium to exist. When the supplier is successful, the supplier's profit will increase when the value of normal cost per unit is as large as is feasible and the value of the reduced cost per unit is as small as is feasible. Decreasing the value of the reduced cost per unit also increases each retailer's profit, while increasing the value of the normal cost per unit decreases each retailer's profit. Setting the pricing parameters for the quantity discount at their extreme boundaries is good for motivating the retailers and achieving maximum profits.

A naïve thought process might suggest that allowing transshipment agreements will lower the volume of items sold and as a result, lower supplier profit. While this is true without quantity discounts, the offering of a quantity discount actually increases profits for both the supplier and the retailers. When quantity discounts are offered, it is evident that there is value to the supplier in allowing limited cooperation among the retailers through transshipment agreements.

The research on a two retailer transshipment system with quantity discounts described in this dissertation is significant to the current literature on transshipment because it shows that the optimal ordering quantities for the retailers have the following properties:

1 Satisfy the first order equilibrium conditions

2 Are not symmetric for the two retailers even though the quantity discount offered to the two retailers by the supplier is identical, all costs are symmetric for the two retailers, and the two retailers are competitive and thus not cooperating in making ordering quantity decisions.

3 Result in improved profits for each retailer over the symmetric equilibrium solutions.

4 Result in improved profits for the supplier.

An important observation in this system is that the methodology in determining the ordering quantities for the retailers results in one retailer acting as a "warehouse".

The “warehouse” retailer orders a large amount of inventory while the other retailer orders a small amount of inventory. The difference between the two ordering quantities becomes even larger as the difference between the normal cost per unit and the reduced cost per unit from the supplier becomes larger. From the supplier’s perspective, controlling the two cost per unit variables allows the supplier to force the two retailers to the two extremes, which results in a larger expected profit for the supplier.

The research on the transshipment system with quantity discounts is further explored through the addition of an additional transshipment partner added to the system. This system involves a third retailer where one retailer can transship with the other two retailers; however, the other two retailers cannot transship with each other. While the methodology remains similar to the two retailer transshipment system, the addition of a third retailer makes the derivation of the expected profit response functions much more complex. The complexity is a result of the transshipment process, the events of which now occur in the three-dimensional space as oppose to the two-dimensional space. While complex, the transshipment event graphs and summing of the pdfs helped verify and validate the transshipment model.

Through analyzing this system with a game theoretic approach, we show that for some cases the following changes occur to the expected profits:

1 Two retailers currently in a transshipment agreement have a decrease in expected profit if either of them takes on another transshipment agreement (going from two to three retailers transshipping).

2 A retailer who previously operated independently as a standard newsvendor where quantity discounts are offered sees an increase in

expected profit by entering in the transshipment agreement with the central retailer.

3 The central retailer has a greater expected profit than each of the two non-central retailers.

4 The supplier has an increase in expected profit.

Consider two retailers that already have a two way transshipment agreement. If a third retailer joins into an transshipment agreement with one of the two original retailers, then both of the retailers currently in a transshipment agreement will have a decrease in expected profit. When the choice to enter into an additional transshipment contract belongs to the retailers and not the supplier, neither of two retailers that currently have a transshipment agreement should enter into a transshipment agreement with another retailer based on the parameters we studied in chapter four. If however one of them does enter into transshipment agreement with another retailer, then the retailer that was not originally part of the transshipment agreement and the supplier would have an increase in expected profit. Thus, the supplier and the independent retailer should be in favor of the additional transshipment agreement as both of them will experience an increase in expected profit.

A comparison is made between value of different quantity discounts schemes to the supplier. Three different quantity discount types are examined when used in a two retailer transshipment system. The three quantity discount types examined are: two-block tariff, two-part tariff, and all-units quantity discounts. Though the quantity discounts share some similarities, it is found that the two-block tariff is the best choice of quantity discount for the supplier to offer to the retailers. The supplier's choice of being able to determine the type of quantity discount offered helps the supplier coordinate activity among a collection of competing, independent, transshipping retailers.

When comparing the two-block tariff quantity discount to the modified two-part tariff quantity discount it is found that for equilibriums that are common between the two quantity discount types, the two quantity discounts result in the same expected retailer and supplier profits. While the two-block tariff quantity discount has the possibility of four different equilibria (A, B, C1, and C2), the two-part tariff quantity discount only has the possibility of two equilibria (C1 and C2). Though the expected profits are the same for the C1 and C2 equilibria between both types, the two-block tariff quantity discount is preferred by the supplier due to the flexibility in stable regions and larger amount of choices for the discount triggering quantity over the two-part tariff quantity discount.

When comparing the two-block tariff quantity discount to the all-units quantity discount it is found that the two-block tariff quantity discount is preferred over the all-units quantity discount because of increased expected profits and flexibility. The all-units quantity discount results in only two equilibria (A and B), while the two-block tariff quantity discount results in the existence of all four equilibria (A, B, C1, and C2) and thus is more flexible then the all-units quantity discount. It is proven that when the retailers are ordering according to Equilibrium B, then the expected supplier profit for the two-block tariff quantity discount case is strictly greater than when the all-units quantity discount is used. The only time that the expected supplier profit between the two cases is equal is when the two retailers are ordering according to Equilibrium A. When the retailers order according to any other equilibrium, the two-block tariff quantity discount either has a greater expected profit for the supplier or the equilibria do not exist in the all-units quantity discount case, in which case the two-block tariff quantity discount has the possibility of their existence.

This research lays the groundwork and gives the fundamental model for future work in the area of transshipments and quantity discounts. An interesting problem in this area would be to determine if there are any other combinations of transshipment partners that result in increases in expected retailer profits as well as expected supplier profits. This leads into another possible problem to solve which is how to resolve the transshipment allocation when two or more retailers have access to the same leftover stock. This problem in particular may lead to an answer to the previous question if the transshipment allocation is unequal and in favor of the retailers previously existing in a transshipment agreement before an additional retailer joins in the agreement.

Another point of interest would be evaluating altered or more complex quantity discount schemes to determine if any particular characteristic about the quantity discounts tends to lead to better results. In particular, a multi-step, two-block tariff quantity discount scheme could lead to interesting results depending upon the amount of transshipment partners in an agreement and how that agreement is set up. It may be possible in a transshipment system with multiple retailers that each retailer could be ordering on different steps of a multi-step, two-block tariff quantity discount scheme. A similar comparison could be made using a multi-step all units quantity discount scheme. Additional points of interest include determining optimal transshipment costs, limiting the system through limiting storage capabilities, and examining the effects of allowing the retailers to refuse to transship. Examining these systems in a transshipment system with quantity discounts could lead to interesting results.

References

Anand, K., Anupindi, R., Bassok, Y. (2008). Strategic Inventories in Vertical Contracts. *Management Science*, 54(10), 1792-1804

Anupindi, R., Bassok, Y., & Zemel, E. (2001). A General Framework for the Study of Decentralized Distribution Systems *Manufacturing Services Oper. Managemen,*, 3(4), 349-368.

Burnetas, A., Gilbert, S., & Smith, C. (2007). Quantity discounts in single-period supply contracts with asymmetric demand information. *IIE Transactions*, *39*(5), 465-479.

Chang, C., Chin, C., & Lin, M. (2006). On the single item multi-supplier system with variable lead-time, price-quantity discount, and resource constraints. *Applied Mathematics and Computation*, *182*(1), 89-97.

Çömez, N., Stecke, K. E., & Çakanyıldırım, M. (2012). In-Season Transshipments Among Competitive Retailers. *Manufacturing & Service Operations Management*, *14*(2), 290-300.

Dolan, R. J. (1987). Quantity Discounts: Managerial Issues and Research Opportunities. *Marketing Science*, *6*(1), 1-22.

Dong, L., & Rudi, N. (2004). Who benefits from transshipment? Exogenous vs. endogenous wholesale prices. *Management Science*, *50*(5), 645-657.

Epstein, Richard A. (2009). *The theory of gambling and statistical logic*. United States: Elsevier Inc.

Fudenberg, D, & Tirole, J. (1991). *Game theory*. Cambridge, MA: MIT Press.

Granot, D., Sošić, G. (2003). A Three-Stage Model for a Decentralized Distribution System of Retailers. *Operations Research*, 51(5), 771-784.

Hanany, E., Tzur, M., & Levran, A. (2010). The Transshipment Fund Mechanism: Coordinating the Decentralized Multilocation Transshipment Problem. *Naval Research Logistics*, *57*(4), 342-353.

Herer, Y. T., & Tzur, M. (2001). The Dynamic Transshipment Problem. *Naval Research Logistics*, *48*(5), 386-408.

Herer, Yale T; Tzur, M. (2003). Optimal and heuristic algorithms for the multi-location dynamic transshipment problem with fixed transshipment costs. *IIE Transactions*, *35*(5), 419-432.

Hu, X., Duenyas, I., & Kapuscinski, R. (2007). Existence of coordinating transshipment prices in a two-location inventory model. *Management Science*, *53*(8), 1289-1302.

Huang, X., & Sošić, G. (2010). Transshipment of inventories: Dual allocations vs. transshipment prices. *Manufacturing & Service Operations Management*, *12*(2), 299-318.

Kokangul, A, & Susuz, Z. (2009). Integrated analytical hierarch process and mathematical programming to supplier selection problem with quantity discount. *Applied Mathematical Modelling*, *33*(3), 1417-1429. Elsevier Inc.

Lee, Y. H., Jung, J. W., & Jeon, Y. S. (2007). An effective lateral transshipment policy to improve service level in the supply chain. *International Journal of Production Economics*, *106*(1), 115-126.

Lien, R. W., Iravani, S. M., Smilowitz, K., & Tzur, M. (2011). An efficient and robust design for transshipment networks. *Production and Operations Management*, *20*(5), 699-713.

Munson, C. L., Hu, J., (2010). Incorporating quantity discounts and their inventory impacts into the centralized purchasing decision. *European Journal of Operational Research*. 201, 581-592.

Nash, J. (1950). The Bargaining Problem. *Econometrica*, *18*(2), 155-162.

Nash, J. (1951). Annals of Mathematics. *Annals of Mathematics*, *54*(2), 286-295.

Nash, J. (1950). Equilibrium Points in N-person Games. *Proceedings of the National Academy of Sciences*, *36*, 48-49.

Olsson, F. (2009). Optimal policies for inventory systems with lateral transshipments. *International Journal of Production Economics*, *118*(1), 175-184.

Olsson, F. (2010). An inventory model with unidirectional lateral transshipments. *European Journal of Operational Research*, *200*(3), 725-732.

Paterson, C., Kiesmüller, G., Teunter, R., & Glazebrook, K. (2011). Inventory models with lateral transshipments: A review. *European Journal of Operational Research*, *210*(2), 125-136.

Rasmusen, Eric. (2007). *Games and information: an introduction to game theory*. Malden, MA: Blackwell publishing.

Rudi, N., Kapur, S., & Pyke, D. F. (2001). A Two-Location Inventory Model with Transshipment and Local Decision Making. *Management Science*, *47*(12), 1668-1680.

Seifert, R. W., Thonemann, U. W., & Sieke, M. A. (2006). Relaxing channel separation: Integrating a virtual store into the supply chain via transshipments. *Iie Transactions*, *38*(11), 917-931.

Slikker, M., Fransoo, J., & Wouters, M. (2005). Cooperation between multiple news-vendors with transshipments. *European Journal of Operational Research*, *167*(2), 370-380.

Shao, J., Krishnan, H., & McCormick, S. T. (2011). Incentives for transshipment in a supply chain with decentralized retailers. *Manufacturing & Service Operations Management*, *13*(3), 361-372.

Shin, H., Benton, W. C., Park, S., (2010). Evaluation of Quantity Discounts for Buyer's Stocking Risk. *International Journal of Management Science*. 16(3) 21-47.

Sošić, G. (2006). Transshipment of Inventories Among Retailers: Myopic vs. Farsighted Stability. *Management Science*, *52*(10), 1493-1508.

Wanke, P. F., & Saliby, E. (2009). Consolidation effects: Whether and how inventories should be pooled. *Transportation Research Part E: Logistics and Transportation Review*, *45*(5), 678-692.

Wong, H., Cattrysse, D., & Oudheusden, D. V. (2005). Inventory pooling of repairable spare parts with non-zero lateral transshipment time and delayed lateral transshipments. *European Journal Of Operational Research*, *165*, 207-218.

Xiao, Y.-bo, Chen, J., Liu, X.-ling, & Zhu, Y. (2008). Dynamic Inventory Transshipment Game between Two Retailers of Seasonal Products. *Systems Engineering - Theory & Practice*, *28*(3), 35-43. Systems Engineering Society of China.

Zhao, H., Deshpande, V., & Ryan, J. K. (2006). Emergency Transshipment in Decentralized Dealer Networks : When to Send and Accept Transshipment Requests. *Naval Research Logistics*, *53*(6), 547-567.

Zhao, X, & Atkins, D. (2009). Transshipment between competing retailers. *IIE Transactions*, *41*(8), 665-676.

Zou, L., Dresner, M., & Windle, R. (2010). Int . J . Production Economics A two-location inventory model with transshipments in a competitive environment. *Intern. Journal of Production Economics*, *125*(2), 235-250. Elsevier.

Appendix A

Expanded Version of π_c^r and π_c^d

$\pi_c^r(Q_c, Q_{nc,i}, Q_{nc,j})$

$$= E\Bigg\{r_c\Bigg(\min(D_c, Q_c)$$

$$+ \min\Bigg[\max(Q_{nc,i}$$

$$- D_{nc,i})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+\right)^+\right]^+\Bigg]$$

$$+ \min\Bigg[\max(Q_{nc,j}$$

$$- D_{nc,j})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,i} - D_{nc,i})^+\right)^+\right]^+\Bigg]\Bigg)$$

$$- c_{ic}\min\Bigg[\max(Q_{nc,i}$$

$$- D_{nc,i})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+\right)^+\right]^+\Bigg]$$

$$- c_{jc}\min\Bigg[\max(Q_{nc,j}$$

$$- D_{nc,j})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,i} - D_{nc,i})^+\right)^+\right]^+\Bigg]$$

$$+ (c_{ci}$$

$$- \tau_{ci})\min\Bigg[\max(D_{nc,i}$$

$$- Q_{nc,i})^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^+\right)^+\right]^+\Bigg]$$

$$+ (c_{cj}$$

$$- \tau_{cj}) \min \Big[\max\big(D_{nc,j}$$

$$- Q_{nc,j}\big)^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max\big(D_{nc,i} - Q_{nc,i}\big)^+\right)^+\right]^+\Big]$$

$$+ s_c \Bigg(\max\,(Q_c - D_c$$

$$- \min \Big[\max\big(D_{nc,i}$$

$$- Q_{nc,i}\big)^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max\big(D_{nc,j} - Q_{nc,j}\big)^+\right)^+\right]^+\Big]$$

$$- \min \Big[\max\big(D_{nc,j}$$

$$- Q_{nc,j}\big)^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max\big(D_{nc,i}$$

$$- Q_{nc,i}\big)^+\right)^+\Big]^+\Big])^+\Bigg)$$

$$- p_c \Bigg(\max\,(D_c - Q_c$$

$$- \min \Big[\max\big(Q_{nc,i}$$

$$- D_{nc,i}\big)^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max\big(Q_{nc,j} - D_{nc,j}\big)^+\right)^+\right]^+\Big]$$

$$- \min \Big[\max\big(Q_{nc,j}$$

$$- D_{nc,j}\big)^+, \max\Big[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max\big(Q_{nc,i}$$

$$- D_{nc,i})^{+}\Big)^{+}\Big]^{+}\Bigg])^{+}\Bigg)\Bigg\} - x(c_n - c_r) - c_r Q_c$$

$\pi_c^d(Q_c, Q_{nc,i}, Q_{nc,j})$

$$= E\left\{r_c\left(\min(D_c, Q_c)\right.\right.$$

$$+ \min\left[\max(Q_{nc,i}\right.$$

$$\left.- D_{nc,i})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+\right)^+\right]^+\right]$$

$$+ \min\left[\max(Q_{nc,j}\right.$$

$$\left.\left.- D_{nc,j})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,i} - D_{nc,i})^+\right)^+\right]^+\right]\right)$$

$$- c_{ic}\min\left[\max(Q_{nc,i}\right.$$

$$\left.- D_{nc,i})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+\right)^+\right]^+\right]$$

$$- c_{jc}\min\left[\max(Q_{nc,j}\right.$$

$$\left.- D_{nc,j})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,i} - D_{nc,i})^+\right)^+\right]^+\right]$$

$$+ (c_{ci}$$

$$- \tau_{ci})\min\left[\max(D_{nc,i}\right.$$

$$\left.- Q_{nc,i})^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^+\right)^+\right]^+\right]$$

$$+ (c_{cj}$$

$$- \tau_{cj}) \min \Big[\max(D_{nc,j}$$

$$- Q_{nc,j})^+, \max \left[\max \left(\frac{Q_c - D_c}{2} \right)^+, \max \left(Q_c - D_c - \max(D_{nc,i} - Q_{nc,i})^+ \right)^+ \right]^+ \Big]$$

$$+ s_c \Big(\max\,(Q_c - D_c$$

$$- \min \Big[\max(D_{nc,i}$$

$$- Q_{nc,i})^+, \max \left[\max \left(\frac{Q_c - D_c}{2} \right)^+, \max \left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^+ \right)^+ \right]^+ \Big]$$

$$- \min \Big[\max(D_{nc,j}$$

$$- Q_{nc,j})^+, \max \Big[\max \left(\frac{Q_c - D_c}{2} \right)^+, \max \Big(Q_c - D_c - \max(D_{nc,i}$$

$$- Q_{nc,i})^+ \Big)^+ \Big]^+ \Big])^+ \Big)$$

$$- p_c \Big(\max\,(D_c - Q_c$$

$$- \min \Big[\max(Q_{nc,i}$$

$$- D_{nc,i})^+, \max \left[\max \left(\frac{D_c - Q_c}{2} \right)^+, \max \left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+ \right)^+ \right]^+ \Big]$$

$$- \min \Big[\max(Q_{nc,j}$$

$$- D_{nc,j})^+, \max \Big[\max \left(\frac{D_c - Q_c}{2} \right)^+, \max \Big(D_c - Q_c - \max(Q_{nc,i}$$

$$- D_{nc,i})^{+}\Big)^{+}\Big]^{+}\Big])^{+}\Bigg)\Bigg\} - c_n Q_c$$

Appendix B

Integral Form of π_c^r and π_c^d

The following equations are taken from a Wolfram Mathematica file. Here the central retailer is Retailer, and the non-central retailers are Retailer 1 and Retailer 3

$$\pi^{r}_{Retailer\ 2} = -\text{CReduced} * \text{q2} + (-\text{CNormal} + \text{CReduced})x$$

$$- c\left(\int_{\text{q2}}^{\text{q1+q2+q3}} \int_{0}^{\frac{1}{2}(-\text{d2+q2})+\text{q3}} \int_{\text{q1}+\frac{1}{2}(-\text{d2+q2})}^{\text{q1}} (-\text{d1}\right.$$

$$+ \text{q1})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d1}\, d\text{d3}\, d\text{d2}$$

$$+ \int_{0}^{\text{q3}} \int_{0}^{\text{d3}} \int_{-\text{d1}-\text{d3}+\text{q1}+\text{q2}+\text{q3}}^{\infty} (-\text{d1} + \text{q1})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d2}\, d\text{d1}\, d\text{d3}$$

$$+ \int_{0}^{\text{q3}} \int_{\text{d3}}^{\text{q1}} \int_{-2\text{d3}+\text{q2}+2\text{q3}}^{\infty} (-\text{d1} + \text{q1})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d2}\, d\text{d1}\, d\text{d3}$$

$$+ \int_{\text{q3}}^{\infty} \int_{0}^{\text{q1}} \int_{-\text{d1}+\text{q1}+\text{q2}}^{\infty} (-\text{d1} + \text{q1})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d2}\, d\text{d1}\, d\text{d3}$$

$$+ \int_{\text{q1}}^{\infty} \int_{0}^{\text{q3}} \int_{\text{q2}}^{-\text{d3}+\text{q2}+\text{q3}} (\text{d2} - \text{q2})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d2}\, d\text{d3}\, d\text{d1}$$

$$+ \frac{1}{2} \int_{\text{q2}}^{\text{q1+q2+q3}} \int_{0}^{\text{q1}+\frac{1}{2}(-\text{d2+q2})} \int_{0}^{\frac{1}{2}(-\text{d2+q2})+\text{q3}} (\text{d2}$$

$$- \text{q2})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d3}\, d\text{d1}\, d\text{d2}$$

$$+ \frac{1}{2} \int_{\text{q2}}^{\text{q1+q2+q3}} \int_{0}^{\frac{1}{2}(-\text{d2+q2})+\text{q3}} \int_{0}^{\text{q1}+\frac{1}{2}(-\text{d2+q2})} (\text{d2}$$

$$- \text{q2})\text{f1}[\text{d1}]\text{f1}[\text{d2}]\text{f1}[\text{d3}]\, d\text{d1}\, d\text{d3}\, d\text{d2}$$

$$+\int_{q3}^{\infty}\int_{0}^{q1}\int_{q2}^{-d1+q1+q2}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+\int_{0}^{q1}\int_{0}^{d1}\int_{-2d1+2q1+q2}^{-d1-d3+q1+q2+q3}(d1+d2-q1$$

$$-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2$$

$$-q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+\int_{0}^{q1}\int_{0}^{d1}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{0}^{q1}\int_{d1}^{q3}\int_{-2d1+2q1+q2}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q1}^{\infty}\int_{0}^{q3}\int_{-d3+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3$$

$$+q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2)$$

$$+ r(\mathrm{q2} \int_0^\infty \int_0^\infty \int_{\mathrm{q2}}^\infty \mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ \int_0^\infty \int_0^\infty \int_0^{\mathrm{q2}} \mathrm{d2f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ \int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_0^{\frac{1}{2}(-\mathrm{d2+q2})+\mathrm{q3}} \int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2+q2})}^{\mathrm{q1}} (-\mathrm{d1}$$

$$+ \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$+ \int_0^{\mathrm{q3}} \int_0^{\mathrm{d3}} \int_{-\mathrm{d1-d3+q1+q2+q3}}^\infty (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$+ \int_0^{\mathrm{q3}} \int_{\mathrm{d3}}^{\mathrm{q1}} \int_{-2\mathrm{d3+q2+2q3}}^\infty (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$+ \int_{\mathrm{q3}}^\infty \int_0^{\mathrm{q1}} \int_{-\mathrm{d1+q1+q2}}^\infty (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$+ \int_{\mathrm{q1}}^\infty \int_0^{\mathrm{q3}} \int_{\mathrm{q2}}^{-\mathrm{d3+q2+q3}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ \frac{1}{2} \int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_0^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2+q2})} \int_0^{\frac{1}{2}(-\mathrm{d2+q2})+\mathrm{q3}} (\mathrm{d2}$$

$$- \mathrm{q2})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d3}\, d\mathrm{d1}\, d\mathrm{d2}$$

$$+\frac{1}{2}\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}}\int_{0}^{\frac{1}{2}(-\mathrm{d2+q2})+\mathrm{q3}}\int_{0}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2+q2})}(\mathrm{d2}$$

$$-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_{\mathrm{q3}}^{\infty}\int_{0}^{\mathrm{q1}}\int_{\mathrm{q2}}^{-\mathrm{d1+q1+q2}}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d1}\,d\mathrm{d3}$$

$$+\int_{0}^{\mathrm{q1}}\int_{0}^{\mathrm{d1}}\int_{-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}}^{-\mathrm{d1-d3+q1+q2+q3}}(\mathrm{d1}+\mathrm{d2}-\mathrm{q1}$$

$$-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d3}\,d\mathrm{d1}$$

$$+\int_{0}^{\mathrm{q3}}\int_{0}^{\mathrm{d3}}\int_{-2\mathrm{d3}+\mathrm{q2}+2\mathrm{q3}}^{-\mathrm{d1-d3+q1+q2+q3}}(\mathrm{d2}+\mathrm{d3}-\mathrm{q2}$$

$$-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d1}\,d\mathrm{d3}$$

$$+\int_{0}^{\mathrm{q1}}\int_{0}^{\mathrm{d1}}\int_{-\mathrm{d1-d3+q1+q2+q3}}^{\infty}(-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d3}\,d\mathrm{d1}$$

$$+\int_{0}^{\mathrm{q1}}\int_{\mathrm{d1}}^{\mathrm{q3}}\int_{-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}}^{\infty}(-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d3}\,d\mathrm{d1}$$

$$+\int_{\mathrm{q1}}^{\infty}\int_{0}^{\mathrm{q3}}\int_{-\mathrm{d3+q2+q3}}^{\infty}(-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d3}\,d\mathrm{d1}$$

$$+\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3$$

$$+q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2)$$

$$+\frac{1}{2}p(\int_{0}^{\infty}\int_{0}^{\infty}\int_{q2}^{\infty}f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1)(2q2$$

$$-2\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}d2f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+2\int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1+\frac{1}{2}(-d2+q2)}^{q1}(-d1$$

$$+q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+2\int_{0}^{q3}\int_{0}^{d3}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{0}^{q3}\int_{d3}^{q1}\int_{-2d3+q2+2q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{q3}^{\infty}\int_{0}^{q1}\int_{-d1+q1+q2}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{q1}^{\infty}\int_{0}^{q3}\int_{q2}^{-d3+q2+q3}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}(d2$$

$$-q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}(d2$$

$$-q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+2\int_{q3}^{\infty}\int_{0}^{q1}\int_{q2}^{-d1+q1+q2}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{0}^{q1}\int_{0}^{d1}\int_{-2d1+2q1+q2}^{-d1-d3+q1+q2+q3}(d1+d2-q1$$

$$-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+2\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2$$

$$-q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{0}^{q1}\int_{0}^{d1}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+2\int_{0}^{q1}\int_{d1}^{q3}\int_{-2d1+2q1+q2}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{q1}^{\infty}\int_{0}^{q3}\int_{-d3+q2+q3}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3$$

$$+ q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2)$$

$$+ s(\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{q2}f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1)(q2$$

$$-\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}d2f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$-\int_{0}^{q2}\int_{0}^{q3}\int_{q1}^{-d2+q1+q2}(d1 - q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1$$

$$- q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_{0}^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1 - q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_{0}^{q2}\int_{0}^{q1}\int_{-d2+q2+q3}^{\infty}(-d2 + q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\frac{1}{2}\int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(-d2$$

$$+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2$$

$$+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{-d1-d2+q1+q2+q3}^{\infty}(-d1-d2+q1$$

$$+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_0^{q1}\int_{q3}^{-d2+q2+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{q3}^{-d1-d2+q1+q2+q3}(d3$$

$$-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$- \int_0^{q2} \int_{q3}^{\frac{1}{2}(-d2+q2)+q3} \int_{-d2-d3+q1+q2+q3}^{\infty} (-d2 - d3 + q2$$

$$+ q3) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2) + (c$$

$$- tau) (\int_0^{q2} \int_0^{q3} \int_{q1}^{-d2+q1+q2} (d1 - q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_{q3}^{\frac{1}{2}(-d2+q2)+q3} \int_{q1}^{-d2-d3+q1+q2+q3} (d1$$

$$- q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_{\frac{1}{2}(-d2+q2)+q3}^{\infty} \int_{q1}^{q1+\frac{1}{2}(-d2+q2)} (d1 - q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_0^{q1} \int_{-d2+q2+q3}^{\infty} (-d2 + q2) f1[d1] f1[d2] f1[d3] \, dd3 \, dd1 \, dd2$$

$$+ \frac{1}{2} \int_0^{q2} \int_{q1+\frac{1}{2}(-d2+q2)}^{\infty} \int_{\frac{1}{2}(-d2+q2)+q3}^{\infty} (-d2$$

$$+ q2) f1[d1] f1[d2] f1[d3] \, dd3 \, dd1 \, dd2$$

$$+ \int_0^{q2} \int_0^{q3} \int_{-d2+q1+q2}^{\infty} (-d2 + q2) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2$$

$$+ q2)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2$$

$$+\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{-d1-d2+q1+q2+q3}^{\infty}(-d1-d2+q1$$

$$+ q2)f1[d1]f1[d2]f1[d3]\,\mathit{d}d3\,\mathit{d}d1\,\mathit{d}d2$$

$$+\int_0^{q2}\int_0^{q1}\int_{q3}^{-d2+q2+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d3\,\mathit{d}d1\,\mathit{d}d2$$

$$+\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{q3}^{-d1-d2+q1+q2+q3}(d3$$

$$- q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d3\,\mathit{d}d1\,\mathit{d}d2$$

$$+\int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d3\,\mathit{d}d1\,\mathit{d}d2$$

$$+\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2-d3+q2$$

$$+ q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2)$$

$$
\begin{aligned}
\pi^{d}_{Retailer\ 2} = & -\mathrm{CNormal} * \mathrm{q2} \\
& - c\Big(\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_{0}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} \int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\mathrm{q1}} (-\mathrm{d1} \\
& + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2} \\
& + \int_{0}^{\mathrm{q3}} \int_{0}^{\mathrm{d3}} \int_{-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3} \\
& + \int_{0}^{\mathrm{q3}} \int_{\mathrm{d3}}^{\mathrm{q1}} \int_{-2\mathrm{d3}+\mathrm{q2}+2\mathrm{q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3} \\
& + \int_{\mathrm{q3}}^{\infty} \int_{0}^{\mathrm{q1}} \int_{-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3} \\
& + \int_{\mathrm{q1}}^{\infty} \int_{0}^{\mathrm{q3}} \int_{\mathrm{q2}}^{-\mathrm{d3}+\mathrm{q2}+\mathrm{q3}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1} \\
& + \frac{1}{2}\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_{0}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} \int_{0}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (\mathrm{d2} \\
& - \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3}\, d\mathrm{d1}\, d\mathrm{d2} \\
& + \frac{1}{2}\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_{0}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} \int_{0}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (\mathrm{d2} \\
& - \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2} \\
& + \int_{\mathrm{q3}}^{\infty} \int_{0}^{\mathrm{q1}} \int_{\mathrm{q2}}^{-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3} \\
& + \int_{0}^{\mathrm{q1}} \int_{0}^{\mathrm{d1}} \int_{-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}}^{-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} (\mathrm{d1} + \mathrm{d2} - \mathrm{q1} - \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}
\end{aligned}
$$

$$+\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2-q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+\int_{0}^{q1}\int_{0}^{d1}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{0}^{q1}\int_{d1}^{q3}\int_{-2d1+2q1+q2}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q1}^{\infty}\int_{0}^{q3}\int_{-d3+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3$$
$$+q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2)$$
$$+r(q2\int_{0}^{\infty}\int_{0}^{\infty}\int_{q2}^{\infty}f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$+\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{q2}d2f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1+\frac{1}{2}(-d2+q2)}^{q1}(-d1$$
$$+q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_{0}^{q3}\int_{0}^{d3}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+\int_{0}^{q3}\int_{d3}^{q1}\int_{-2d3+q2+2q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+\int_{q3}^{\infty}\int_{0}^{q1}\int_{-d1+q1+q2}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+\int_{q1}^{\infty}\int_{0}^{q3}\int_{q2}^{-d3+q2+q3}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$+\frac{1}{2}\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}(d2$$
$$-q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$
$$+\frac{1}{2}\int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}(d2$$
$$-q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$
$$+\int_{q3}^{\infty}\int_{0}^{q1}\int_{q2}^{-d1+q1+q2}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$
$$+\int_{0}^{q1}\int_{0}^{d1}\int_{-2d1+2q1+q2}^{-d1-d3+q1+q2+q3}(d1+d2-q1-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$+\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2-q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$
$$+\int_{0}^{q1}\int_{0}^{d1}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$+\int_{0}^{q1}\int_{d1}^{q3}\int_{-2d1+2q1+q2}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$+\int_{q1}^{\infty}\int_{0}^{q3}\int_{-d3+q2+q3}^{\infty}(-d3+q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$+\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3$$
$$+q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2)$$

$$+\frac{1}{2}p(\int_0^\infty\int_0^\infty\int_{q2}^\infty f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1)(2q2$$

$$-2\int_0^\infty\int_0^\infty\int_0^\infty d2f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+2\int_{q2}^{q1+q2+q3}\int_0^{\frac{1}{2}(-d2+q2)+q3}\int_{q1+\frac{1}{2}(-d2+q2)}^{q1}(-d1$$

$$+q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+2\int_0^{q3}\int_0^{d3}\int_{-d1-d3+q1+q2+q3}^\infty(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_0^{q3}\int_{d3}^{q1}\int_{-2d3+q2+2q3}^\infty(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{q3}^\infty\int_0^{q1}\int_{-d1+q1+q2}^\infty(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+2\int_{q1}^\infty\int_0^{q3}\int_{q2}^{-d3+q2+q3}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+\int_{q2}^{q1+q2+q3}\int_0^{q1+\frac{1}{2}(-d2+q2)}\int_0^{\frac{1}{2}(-d2+q2)+q3}(d2$$

$$-q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_{q2}^{q1+q2+q3}\int_0^{\frac{1}{2}(-d2+q2)+q3}\int_0^{q1+\frac{1}{2}(-d2+q2)}(d2$$

$$-q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+2\int_{q3}^\infty\int_0^{q1}\int_{q2}^{-d1+q1+q2}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+ 2\int_{0}^{q1}\int_{0}^{d1}\int_{-2d1+2q1+q2}^{-d1-d3+q1+q2+q3}(d1 + d2 - q1 - q2)f1[d1]f1[d2]f1[d3]\, dd2\, dd3\, dd1$$

$$+ 2\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2 + d3 - q2 - q3)f1[d1]f1[d2]f1[d3]\, dd2\, dd1\, dd3$$

$$+ 2\int_{0}^{q1}\int_{0}^{d1}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\, dd2\, dd3\, dd1$$

$$+ 2\int_{0}^{q1}\int_{d1}^{q3}\int_{-2d1+2q1+q2}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\, dd2\, dd3\, dd1$$

$$+ 2\int_{q1}^{\infty}\int_{0}^{q3}\int_{-d3+q2+q3}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\, dd2\, dd3\, dd1$$

$$+ 2\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3 + q3)f1[d1]f1[d2]f1[d3]\, dd3\, dd1\, dd2)$$

$$+ s(\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{q2}f1[d1]f1[d2]f1[d3]\, dd2\, dd3\, dd1)(q2$$

$$- \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}d2f1[d1]f1[d2]f1[d3]\, dd2\, dd3\, dd1$$

$$- \int_{0}^{q2}\int_{0}^{q3}\int_{q1}^{-d2+q1+q2}(d1 - q1)f1[d1]f1[d2]f1[d3]\, dd1\, dd3\, dd2$$

$$- \int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1 - q1)f1[d1]f1[d2]f1[d3]\, dd1\, dd3\, dd2$$

$$-\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_0^{q2}\int_0^{q1}\int_{-d2+q2+q3}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\frac{1}{2}\int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{-d1-d2+q1+q2+q3}^{\infty}(-d1-d2+q1$$
$$+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_0^{q1}\int_{q3}^{-d2+q2+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{q3}^{-d1-d2+q1+q2+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$-\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2-d3+q2$$
$$+q3)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)+(c$$

$$- \text{tau})\left(\int_0^{q2}\int_0^{q3}\int_{q1}^{-d2+q1+q2}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2\right.$$

$$+\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_0^{q1}\int_{-d2+q2+q3}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\frac{1}{2}\int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{-d1-d2+q1+q2+q3}^{\infty}(-d1-d2+q1$$

$$+q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_0^{q2}\int_0^{q1}\int_{q3}^{-d2+q2+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{q3}^{-d1-d2+q1+q2+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_{0}^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d3-q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+\int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2-d3+q2$$

$$+q3)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)$$

Appendix C

Expanded Version of $\pi_{nc,i}^{r}$ and $\pi_{nc,i}^{d}$

$\pi_{nc,i}^{r}(Q_c, Q_{nc,i}, Q_{nc,j})$

$$= E\Bigg\{r_{nc,i} \min(D_{nc,i}, Q_{nc,i})$$

$$+ \min\Bigg[\max(D_{nc,i}$$

$$- Q_{nc,i})^{+}, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^{+}, \max\left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^{+}\right)^{+}\right]^{+}\Bigg]$$

$$- c_{ci} \min\Bigg[\max(D_{nc,i}$$

$$- Q_{nc,i})^{+}, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^{+}, \max\left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^{+}\right)^{+}\right]^{+}\Bigg]$$

$$+ (c_{ic}$$

$$- \tau_{ic}) \min\Bigg[\max(Q_{nc,i}$$

$$- D_{nc,i})^{+}, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^{+}, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^{+}\right)^{+}\right]^{+}\Bigg]$$

$$+ s_{nc,i} \max(Q_{nc,i} - D_{nc,i}$$

$$- \min\Bigg[\max(Q_{nc,i}$$

$$- D_{nc,i})^{+}, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^{+}, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^{+}\right)^{+}\right]^{+}\Bigg])^{+}$$

$$- p_{nc,i} \max(D_{nc,i} - Q_{nc,i}$$

$$- \min\Bigg[\max(D_{nc,i}$$

$$- Q_{nc,i})^{+}, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^{+}, \max\left(Q_c - D_c - \max(D_{nc,j}\right.\right.$$

$$\left.\left.- Q_{nc,j})^{+}\right)^{+}\right]^{+}\Big])^{+}\Big\} - x(c_n - c_r) - c_r Q_{nc,i}$$

$\pi^{d}_{nc,i}(Q_c, Q_{nc,i}, Q_{nc,j})$

$$= E\left\{r_{nc,i} \min(D_{nc,i}, Q_{nc,i})\right.$$

$$+ \min\left[\max(D_{nc,i}\right.$$

$$\left.- Q_{nc,i})^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^+\right)^+\right]^+\right]$$

$$- c_{ci} \min\left[\max(D_{nc,i}\right.$$

$$\left.- Q_{nc,i})^+, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^+, \max\left(Q_c - D_c - \max(D_{nc,j} - Q_{nc,j})^+\right)^+\right]^+\right]$$

$$+ (c_{ic}$$

$$- \tau_{ic}) \min\left[\max(Q_{nc,i}\right.$$

$$\left.- D_{nc,i})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+\right)^+\right]^+\right]$$

$$+ s_{nc,i} \max (Q_{nc,i} - D_{nc,i}$$

$$- \min\left[\max(Q_{nc,i}\right.$$

$$\left.- D_{nc,i})^+, \max\left[\max\left(\frac{D_c - Q_c}{2}\right)^+, \max\left(D_c - Q_c - \max(Q_{nc,j} - D_{nc,j})^+\right)^+\right]^+\right])^+$$

$$- p_{nc,i} \max (D_{nc,i} - Q_{nc,i}$$

$$- \min\left[\max(D_{nc,i}\right.$$

$$- Q_{nc,i})^{+}, \max\left[\max\left(\frac{Q_c - D_c}{2}\right)^{+}, \max\left(Q_c - D_c - \max\left(D_{nc,j}\right.\right.\right.$$

$$\left.\left.\left.- Q_{nc,j})^{+}\right)^{+}\right]^{+}\right])^{+}\Bigg\} - c_n Q_{nc,i}$$

Appendix D

Integral Form of $\boldsymbol{\pi^{r}_{nc,i}}$ and $\boldsymbol{\pi^{d}_{nc,i}}$

The following equations are taken from a Wolfram Mathematica file. Here the central retailer is Retailer 2, and the non-central retailers are Retailer 1 and Retailer 3.

$$\pi^{r}_{Retailer\ 1} = -\mathrm{CReduced} * \mathrm{q1} - (\mathrm{CNormal} - \mathrm{CReduced})x$$

$$+ s\left(\int_0^{\infty}\int_0^{\infty}\int_0^{\mathrm{q1}} \mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}\right)(\mathrm{q1}$$

$$- \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{d1f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$- \int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}}\int_0^{\frac{1}{2}(\mathrm{-d2+q2})+\mathrm{q3}}\int_{\mathrm{q1}+\frac{1}{2}(\mathrm{-d2+q2})}^{\mathrm{q1}} (-\mathrm{d1}$$

$$+ \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$- \int_0^{\mathrm{q3}}\int_0^{\mathrm{d3}}\int_{\mathrm{-d1-d3+q1+q2+q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \int_0^{\mathrm{q3}}\int_{\mathrm{d3}}^{\mathrm{q1}}\int_{\mathrm{-2d3+q2+2q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \int_{\mathrm{q3}}^{\infty}\int_0^{\mathrm{q1}}\int_{\mathrm{-d1+q1+q2}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \frac{1}{2}\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}}\int_0^{\frac{1}{2}(\mathrm{-d2+q2})+\mathrm{q3}}\int_0^{\mathrm{q1}+\frac{1}{2}(\mathrm{-d2+q2})} (\mathrm{d2}$$

$$- \mathrm{q2})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$- \int_{\mathrm{q3}}^{\infty}\int_0^{\mathrm{q1}}\int_{\mathrm{q2}}^{\mathrm{-d1+q1+q2}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \int_0^{q3} \int_0^{d3} \int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3} (d2 + d3 - q2$$

$$- q3) f1[d1] f1[d2] f1[d3] \, dd2 \, dd1 \, dd3) + (c$$

$$- tau)(\int_{q2}^{q1+q2+q3} \int_0^{\frac{1}{2}(-d2+q2)+q3} \int_{q1+\frac{1}{2}(-d2+q2)}^{q1} (-d1$$

$$+ q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q3} \int_0^{d3} \int_{-d1-d3+q1+q2+q3}^{\infty} (-d1 + q1) f1[d1] f1[d2] f1[d3] \, dd2 \, dd1 \, dd3$$

$$+ \int_0^{q3} \int_{d3}^{q1} \int_{-2d3+q2+2q3}^{\infty} (-d1 + q1) f1[d1] f1[d2] f1[d3] \, dd2 \, dd1 \, dd3$$

$$+ \int_{q3}^{\infty} \int_0^{q1} \int_{-d1+q1+q2}^{\infty} (-d1 + q1) f1[d1] f1[d2] f1[d3] \, dd2 \, dd1 \, dd3$$

$$+ \frac{1}{2} \int_{q2}^{q1+q2+q3} \int_0^{\frac{1}{2}(-d2+q2)+q3} \int_0^{q1+\frac{1}{2}(-d2+q2)} (d2$$

$$- q2) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_{q3}^{\infty} \int_0^{q1} \int_{q2}^{-d1+q1+q2} (d2 - q2) f1[d1] f1[d2] f1[d3] \, dd2 \, dd1 \, dd3$$

$$+\int_0^{q3}\int_0^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2$$

$$-q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3)$$

$$-p(\int_0^{\infty}\int_0^{\infty}\int_{q1}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)(-q1$$

$$+\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}d1f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$-\int_0^{q2}\int_0^{q3}\int_{q1}^{-d2+q1+q2}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1$$

$$-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2$$

$$+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- \int_0^{q2} \int_{q3}^{\frac{1}{2}(-d2+q2)+q3} \int_{-d2-d3+q1+q2+q3}^{\infty} (-d2 - d3 + q2$$

$$+ q3) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2)$$

$$- c(\int_0^{q2} \int_0^{q3} \int_{q1}^{-d2+q1+q2} (d1 - q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_{q3}^{\frac{1}{2}(-d2+q2)+q3} \int_{q1}^{-d2-d3+q1+q2+q3} (d1$$

$$- q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_{\frac{1}{2}(-d2+q2)+q3}^{\infty} \int_{q1}^{q1+\frac{1}{2}(-d2+q2)} (d1 - q1) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_0^{q3} \int_{-d2+q1+q2}^{\infty} (-d2 + q2) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \frac{1}{2} \int_0^{q2} \int_{\frac{1}{2}(-d2+q2)+q3}^{\infty} \int_{q1+\frac{1}{2}(-d2+q2)}^{\infty} (-d2$$

$$+ q2) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2$$

$$+ \int_0^{q2} \int_{q3}^{\frac{1}{2}(-d2+q2)+q3} \int_{-d2-d3+q1+q2+q3}^{\infty} (-d2 - d3 + q2$$

$$+ q3) f1[d1] f1[d2] f1[d3] \, dd1 \, dd3 \, dd2)$$

$$+ r\Big(\mathrm{q1}\int_0^\infty\int_0^\infty\int_{\mathrm{q1}}^\infty \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_0^\infty\int_0^\infty\int_0^{\mathrm{q1}} \mathrm{d1}\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}\int_0^{\mathrm{q3}}\int_{\mathrm{q1}}^{-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}} (\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}\int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}\int_{\mathrm{q1}}^{-\mathrm{d2}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} (\mathrm{d1}$$

$$-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty}\int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}\int_0^{\mathrm{q3}}\int_{-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}}^{\infty} (-\mathrm{d2}+\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\frac{1}{2}\int_0^{\mathrm{q2}}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty}\int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\infty} (-\mathrm{d2}$$

$$+\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}\int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}\int_{-\mathrm{d2}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}^{\infty} (-\mathrm{d2}-\mathrm{d3}+\mathrm{q2}$$

$$+\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3}\,d\mathrm{d2}\Big)$$

$$\pi^{d}_{Retailer\ 1} = -\mathrm{CNormal} * \mathrm{q1} + s\Big(\int_0^\infty \int_0^\infty \int_0^{\mathrm{q1}} \mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}\Big)(\mathrm{q1}$$

$$- \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d1f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$- \int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_0^{\frac{1}{2}(-\mathrm{d2+q2})+\mathrm{q3}} \int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2+q2})}^{\mathrm{q1}} (-\mathrm{d1}$$

$$+ \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$- \int_0^{\mathrm{q3}} \int_0^{\mathrm{d3}} \int_{-\mathrm{d1-d3+q1+q2+q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \int_0^{\mathrm{q3}} \int_{\mathrm{d3}}^{\mathrm{q1}} \int_{-2\mathrm{d3+q2+2q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \int_{\mathrm{q3}}^{\infty} \int_0^{\mathrm{q1}} \int_{-\mathrm{d1+q1+q2}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \frac{1}{2} \int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_0^{\frac{1}{2}(-\mathrm{d2+q2})+\mathrm{q3}} \int_0^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2+q2})} (\mathrm{d2}$$

$$- \mathrm{q2})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$- \int_{\mathrm{q3}}^{\infty} \int_0^{\mathrm{q1}} \int_{\mathrm{q2}}^{-\mathrm{d1+q1+q2}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$- \int_0^{\mathrm{q3}} \int_0^{\mathrm{d3}} \int_{-2\mathrm{d3+q2+2q3}}^{-\mathrm{d1-d3+q1+q2+q3}} (\mathrm{d2} + \mathrm{d3} - \mathrm{q2} - \mathrm{q3})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3})$$

$$+ (c$$

$$- \mathrm{tau})\Big(\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}} \int_0^{\frac{1}{2}(-\mathrm{d2+q2})+\mathrm{q3}} \int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2+q2})}^{\mathrm{q1}} (-\mathrm{d1}$$

$$+ \mathrm{q1})\mathrm{f1[d1]f1[d2]f1[d3]}\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$+\int_0^{q3}\int_0^{d3}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$
$$+\int_0^{q3}\int_{d3}^{q1}\int_{-2d3+q2+2q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$
$$+\int_{q3}^{\infty}\int_0^{q1}\int_{-d1+q1+q2}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$
$$+\frac{1}{2}\int_{q2}^{q1+q2+q3}\int_0^{\frac{1}{2}(-d2+q2)+q3}\int_0^{q1+\frac{1}{2}(-d2+q2)}(d2$$
$$-q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$
$$+\int_{q3}^{\infty}\int_0^{q1}\int_{q2}^{-d1+q1+q2}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$
$$+\int_0^{q3}\int_0^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2-q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3)$$
$$-p(\int_0^{\infty}\int_0^{\infty}\int_{q1}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)(-q1$$
$$+\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}d1f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$
$$-\int_0^{q2}\int_0^{q3}\int_{q1}^{-d2+q1+q2}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$
$$-\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$
$$-\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$
-\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
-\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
-\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2-d3+q2
$$

$$
+q3)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)
$$

$$
-c(\int_0^{q2}\int_0^{q3}\int_{q1}^{-d2+q1+q2}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
+\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
+\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
+\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
+\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
+\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2-d3+q2
$$

$$
+q3)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)
$$

$$
+r(q1\int_0^{\infty}\int_0^{\infty}\int_{q1}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$
+\int_0^{\infty}\int_0^{\infty}\int_0^{q1}d1f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2
$$

$$+\int_0^{q2}\int_0^{q3}\int_{q1}^{-d2+q1+q2}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\frac{1}{2}\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+\int_0^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2-d3+q2$$

$$+q3)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)$$

Appendix E

Three Retailer Transshipment Event Graph Breakdown

The figures in this appendix show the breakdown of the transshipment events in the three retailer transshipment system. In the figures in this appendix, the central retailer is Retailer 2 and the non-central retailers are Retailer 1 and Retailer 3.

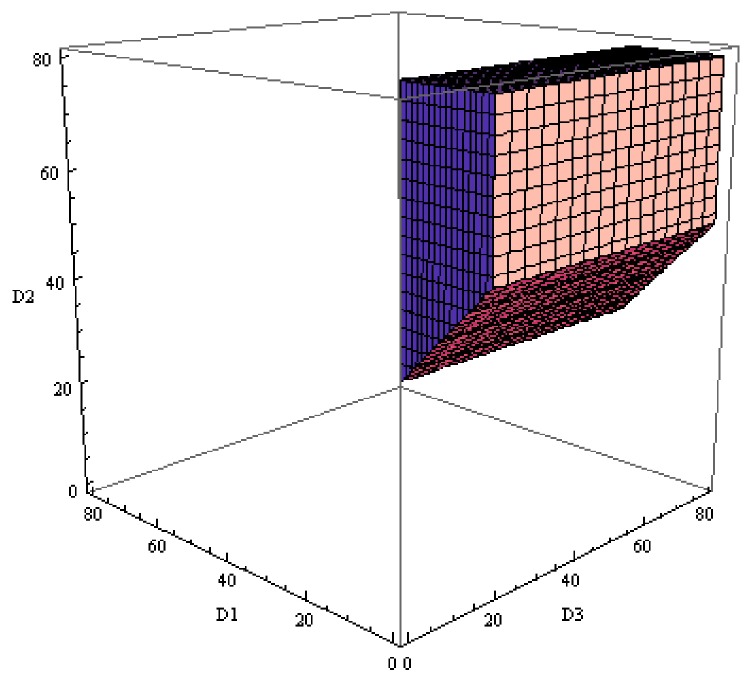

Figure 52: Excess demand for Retailer 2 and Retailer 3. Excess inventory for Retailer 1. Transshipment from Retailer 1 to Retailer 2 in the amount of $Q_1 - D_1$.

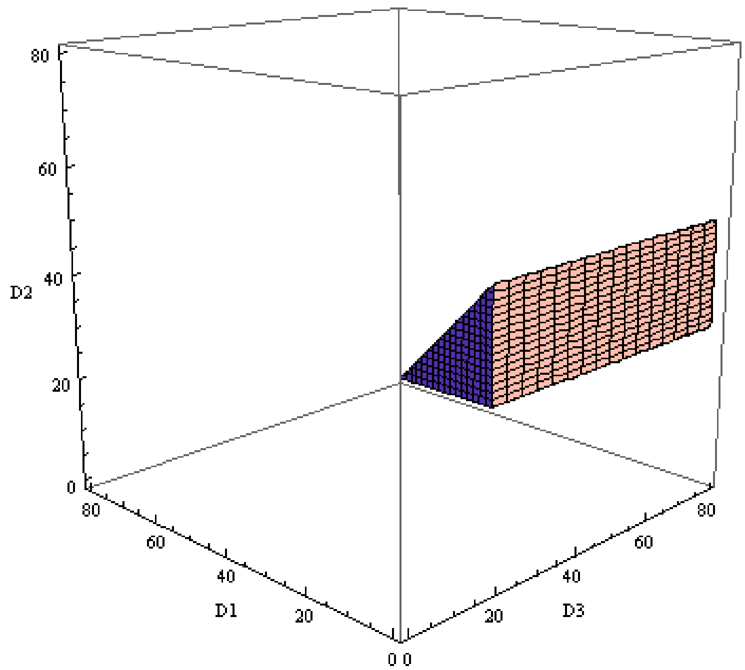

Figure 53: Excess demand for Retailer 2 and Retailer 3. Excess inventory for Retailer 1. Transshipment from Retailer 1 to Retailer 2 in the amount of $D_2 - Q_2$.

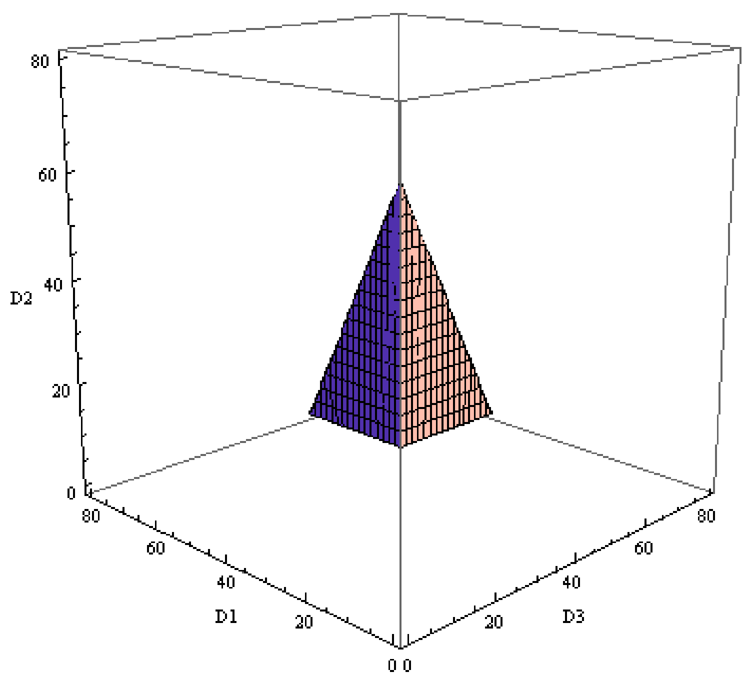

Figure 54: Excess demand for Retailer 2. Excess inventory for Retailer 1 and Retailer 3. Transshipment from Retailer 1 to Retailer 2 in the amount of $D_2 - Q_2/_2$.

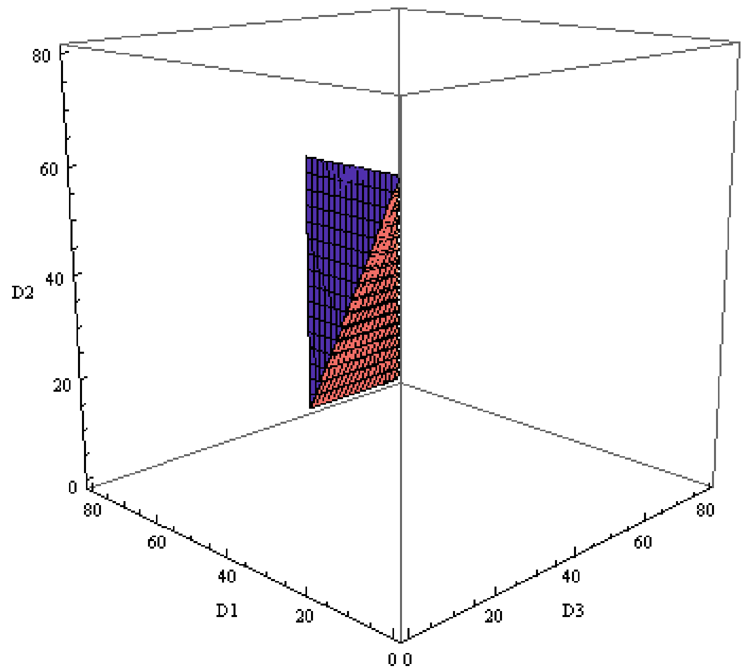

Figure 55: Excess demand for Retailer 2. Excess inventory for Retailer 1 and Retailer 3. Transshipment from Retailer 1 to Retailer 2 in the amount of $Q_1 - D_1$.

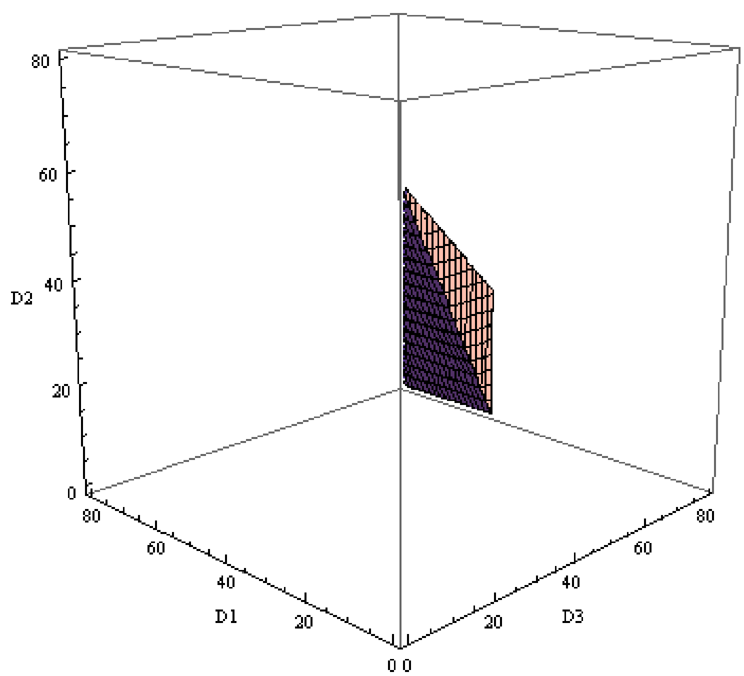

Figure 56: Excess demand for Retailer 2. Excess inventory for Retailer 1 and Retailer 3. Transshipment from Retailer 1 to Retailer 2 in the amount of $D_2 - Q_2 - Q_3 + D_3$

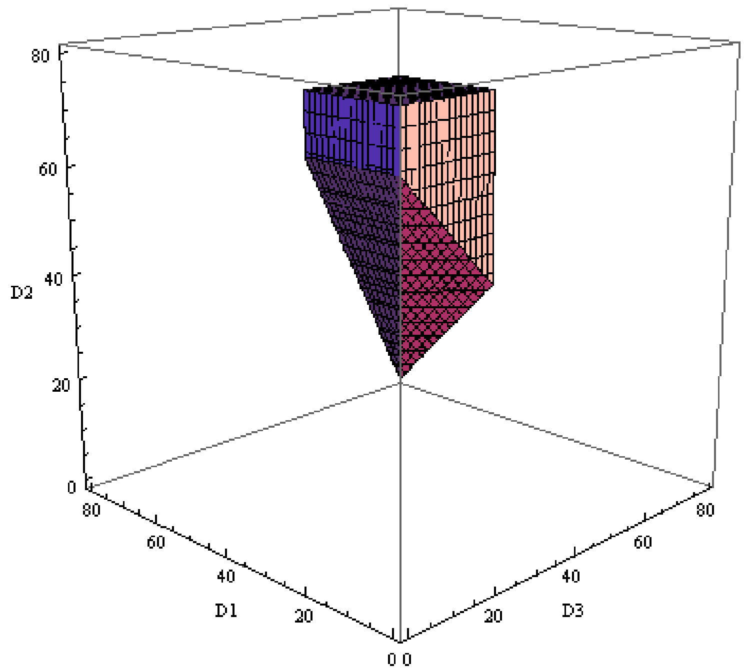

Figure 57: Excess demand for Retailer 2. Excess inventory for Retailer 1 and Retailer 3. Transshipment from Retailer 1 to Retailer 2 in the amount of $Q_1 - D_1$.

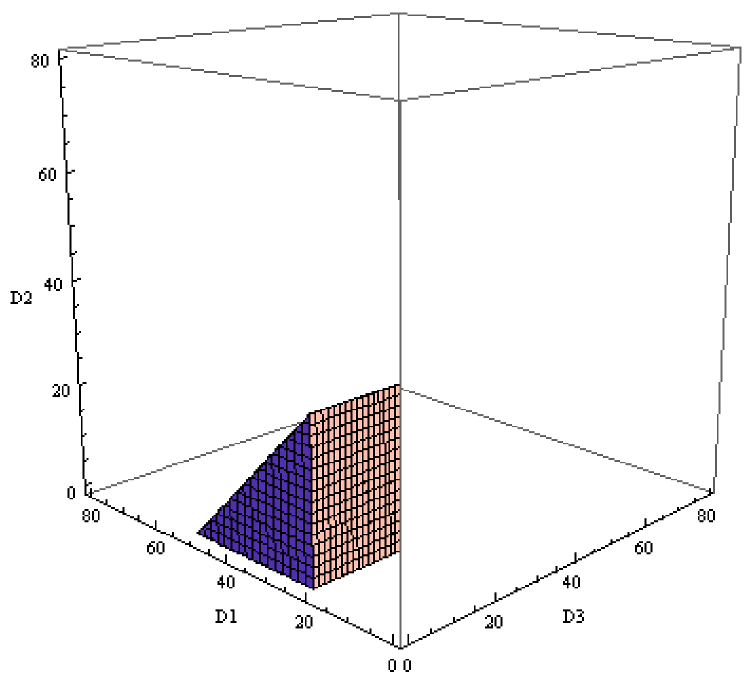

Figure 58: Excess inventory for Retailer 2 and Retailer 3. Excess demand for Retailer 1. Transshipment from Retailer 2 to Retailer 1 in the amount of $D_1 - Q_1$.

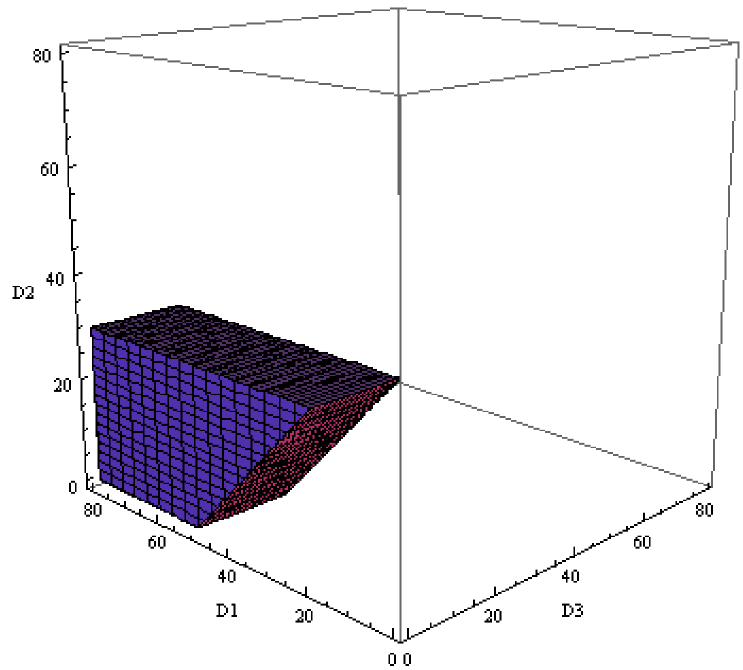

Figure 59: Excess inventory for Retailer 2 and Retailer 3. Excess demand for Retailer 1. Transshipment from Retailer 2 to Retailer 1 in the amount of $Q_2 - D_2$.

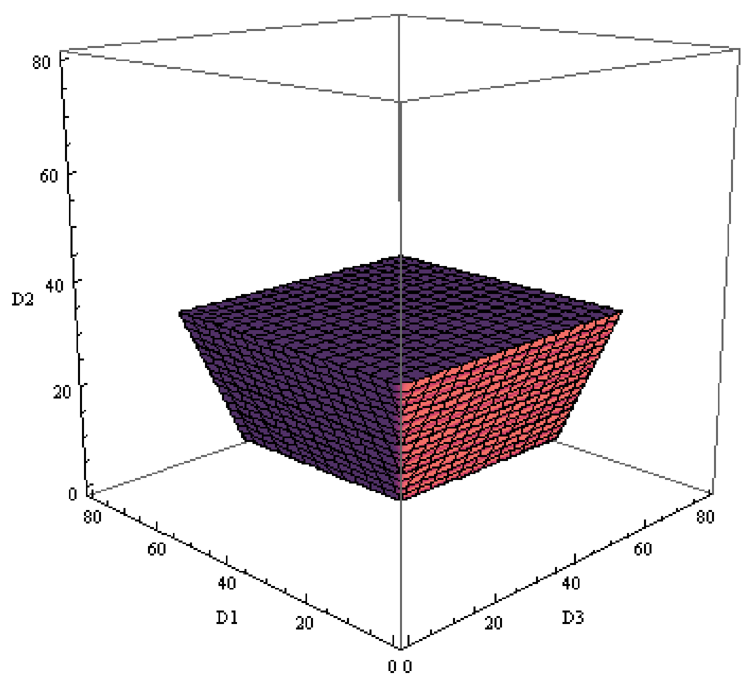

Figure 60: Excess inventory for Retailer 2. Excess demand for Retailer 1 and Retailer 3. Transshipment from Retailer 2 to Retailer 1 in the amount of $Q_2 - D_2/_2$.

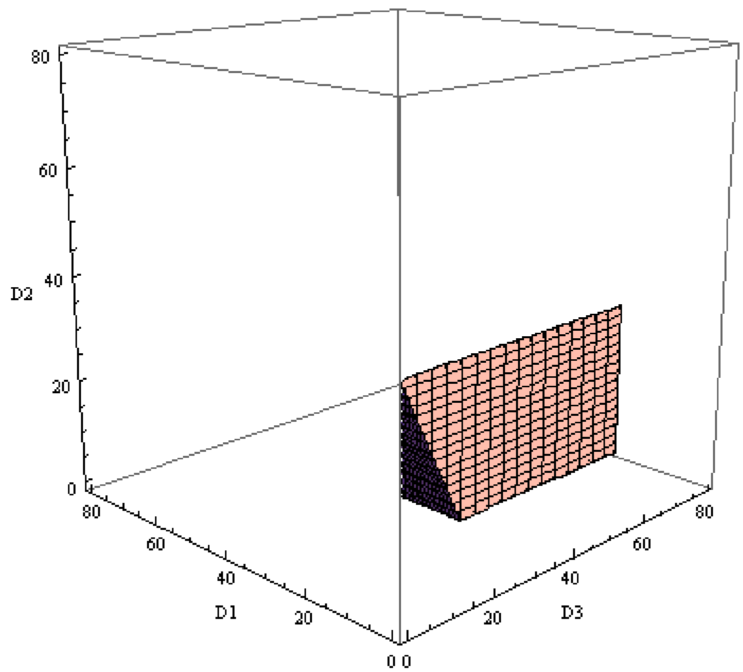

Figure 61: Excess inventory for Retailer 2. Excess demand for Retailer 1 and Retailer 3. Transshipment from Retailer 2 to Retailer 1 in the amount of $D_1 - Q_1$.

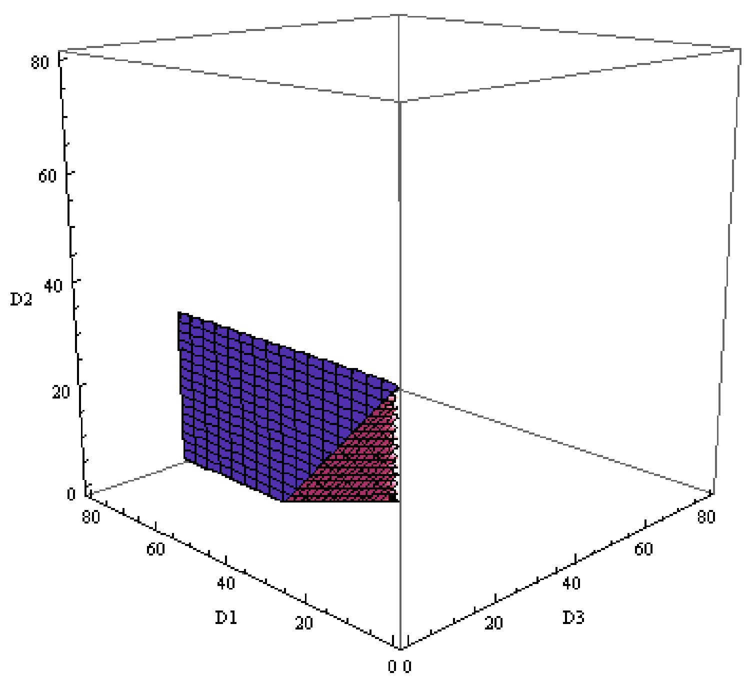

Figure 62: Excess inventory for Retailer 2. Excess demand for Retailer 1 and Retailer 3. Transshipment from Retailer 2 to Retailer 1 in the amount of $Q_2 - D_2 - D_3 + Q_3$.

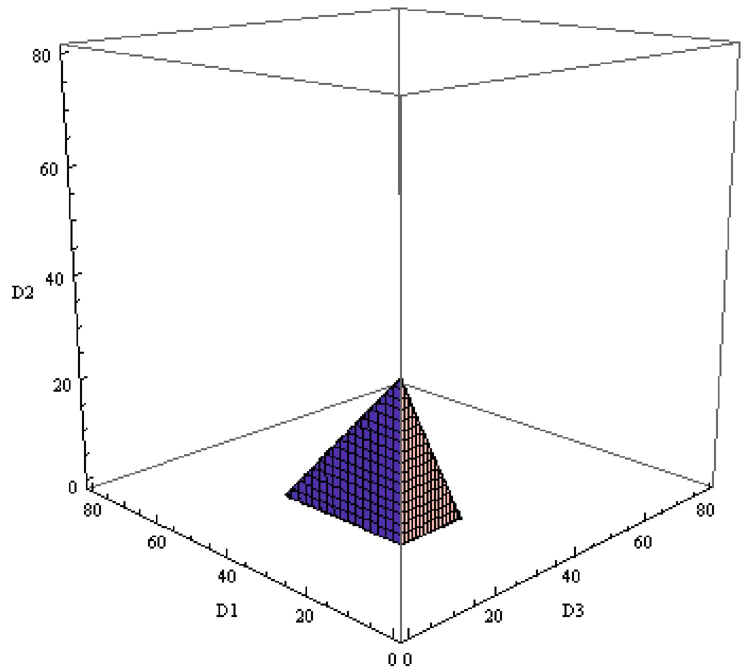

Figure 63: Excess inventory for Retailer 2. Excess demand for Retailer 1 and Retailer 3. Transshipment from Retailer 2 to Retailer 1 in the amount of $D_1 - Q_1$.

Appendix F

Summation of PDFs in Three Retailer System

This appendix gives the Wolfram Mathematica document showing the summation of the pdfs in the three retailer system.

```
|· This file shows the pdfs that make up the four transshipment operators: T12,
T32, T21, and T23. In certain regions of the transshipment operators more than
 one transshipment takes place while in other regions only one transshipment takes
 place. In order for simplicity ·|

Clear| f1, x, lambda, q1, q2, q3|

|· Distribution ·|
f1| x_| :: lambda· Exp| · lambda· x|
lambda : 1| 20;

q2 : 30;
q1 : 20;
q3 : q1;

|· Defining T12, T13, T21, T23 ·|
T12pdfs| q1_, q2_, q3_| ::
 Integrate| f1| d1| · f1| d2| · f1| d3| , | d3, q3, Infinity| , | d1, 0, q1| , | d2, q1· q2· d1, Infinity| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d3, q3, Infinity| , | d1, 0, q1| , | d2, q2, q1· q2· d1| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d2, q2, q2· q1· q3| , | d3, 0, | q2· d2| | 2· q3| ,
   | d1, 0, | q2· d2| | 2· q1| | · Integrate| f1| d1| · f1| d2| · f1| d3| , | d2, q2, q2· q1· q3| ,
   | d3, 0, | q2· d2| | 2· q3| , | d1, | q2· d2| | 2· q1, q1| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d3, 0, q3| , | d1, 0, d3| ,
   | d2, 2 q3· q2· 2 d3, q1· q2· q3· d1· d3| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d3, 0, q3| , | d1, d3, q1| , | d2, q2· 2 q3· 2· d3, Infinity| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d3, 0, q3| , | d1, 0, d3| , | d2, q1· q2· q3· d1· d3, Infinity| |

T32pdfs| q1_, q2_, q3_| ::
 Integrate| f1| d1| · f1| d2| · f1| d3| , | d1, q1, Infinity| , | d3, 0, q3| , | d2, q3· q2· d3, Infinity| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d1, q1, Infinity| , | d3, 0, q3| , | d2, q2, q3· q2· d3| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d2, q2, q2· q1· q3| , | d1, 0, | q2· d2| | 2· q1| ,
   | d3, 0, | q2· d2| | 2· q3| | · Integrate| f1| d1| · f1| d2| · f1| d3| , | d2, q2, q2· q1· q3| ,
   | d1, 0, | q2· d2| | 2· q1| , | d3, | q2· d2| | 2· q3, q3| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d1, 0, q1| , | d3, 0, d1| ,
   | d2, 2 q1· q2· 2 d1, q1· q2· q3· d1· d3| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d1, 0, q1| , | d3, d1, q3| , | d2, q2· 2 q1· 2· d1, Infinity| | ·
  Integrate| f1| d1| · f1| d2| · f1| d3| , | d1, 0, q1| , | d3, 0, d1| , | d2, q1· q2· q3· d1· d3, Infinity| |
```

```
T21pdfs[q1_, q2_, q3_] :=
 Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, 0, q3}, {d1, q1, q2 - d2 + q1}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, 0, q3}, {d1, q1 + q2 - d2, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, (q2 - d2) / 2 + q3, Infinity},
   {d1, (q2 - d2) / 2 + q1, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, (q2 - d2) / 2 + q3, Infinity},
   {d1, q1, (q2 - d2) / 2 + q1}] + Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2},
   {d3, q3, (q2 - d2) / 2 + q3}, {d1, q1 + q2 + q3 - d2 - d3, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, q3, (q2 - d2) / 2 + q3},
   {d1, q1, q1 + q2 + q3 - d2 - d3}]

T23pdfs[q1_, q2_, q3_] :=
 Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, 0, q1}, {d3, q3, q2 - d2 + q3}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, 0, q1}, {d3, q3 + q2 - d2, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, (q2 - d2) / 2 + q1, Infinity},
   {d3, (q2 - d2) / 2 + q3, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, (q2 - d2) / 2 + q1, Infinity},
   {d3, q3, (q2 - d2) / 2 + q3}] + Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2},
   {d1, q1, (q2 - d2) / 2 + q1}, {d3, q1 + q2 + q3 - d2 - d1, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, q1, (q2 - d2) / 2 + q1},
   {d3, q3, q1 + q2 + q3 - d2 - d1}]

Zeropdfs[q1_, q2_, q3_] := Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, 0, q1}, {d3, 0, q3}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, q2, Infinity}, {d1, q1, Infinity}, {d3, q3, Infinity}]

Singlepdfs[q1_, q2_, q3_] :=
 Integrate[f1[d1] * f1[d2] * f1[d3], {d3, q3, Infinity}, {d1, 0, q1}, {d2, q1 + q2 - d1, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d3, q3, Infinity}, {d1, 0, q1}, {d2, q2, q1 + q2 - d1}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d1, q1, Infinity}, {d3, 0, q3}, {d2, q3 + q2 - d3, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d1, q1, Infinity}, {d3, 0, q3}, {d2, q2, q3 + q2 - d3}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, 0, q3}, {d1, q1, q2 - d2 + q1}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d3, 0, q3}, {d1, q1 + q2 - d2, Infinity}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, 0, q1}, {d3, q3, q2 - d2 + q3}] +
  Integrate[f1[d1] * f1[d2] * f1[d3], {d2, 0, q2}, {d1, 0, q1}, {d3, q3 + q2 - d2, Infinity}] +
  Zeropdfs[q1, q2, q3]
```

```
Singlepdfs[q1, q2, q3] // N
```

0.805705

```
(T23pdfs[q1, q2, q3] + T21pdfs[q1, q2, q3] + T12pdfs[q1, q2, q3] + T32pdfs[q1, q2, q3] +
  Zeropdfs[q1, q2, q3] + Singlepdfs[q1, q2, q3])
```

$$\frac{2}{\ {}^{9/2}} + \frac{12}{\ {}^{7/2}} + \frac{-7 + 4\ {}^{3/4}}{\ {}^{7/2}} + \frac{4(-2 + \ )}{\ {}^{7/2}} + \frac{6(-1 + \ )}{\ {}^{7/2}} - \frac{4}{\ {}^{11/4}} + \frac{2}{\ {}^{2}} + \frac{-5 + 2\ }{\ {}^{7/2}} +$$

$$\frac{2\left(5 + \left(-5 + 2(-1 + \ )\sqrt{\ }\right)\ \right)}{\ {}^{7/2}} + \frac{2(-1 + \ )^{2}(-1 + \ )^{3/2}}{\ {}^{7/2}} + \frac{2\left(5 - 4\ + \ ^{2}\right)}{\ {}^{7/2}} + \frac{2\left(-1 - 2\ + \ ^{2}\right)}{\ {}^{9/2}}$$

```
Simplify[%]
```

2

Appendix G

Definition of $k_c(Q_c, Q_{nc,i}, Q_{nc,j})$

The following equation is taken from a Wolfram Mathematica file. Here the central retailer is Retailer 2, and the non-central retailers are Retailer 1 and Retailer 3.

$\boldsymbol{k}_{\boldsymbol{Retailer\ 2}}(\boldsymbol{Q}_{\boldsymbol{Retailer\ 2}}, \boldsymbol{Q}_{\boldsymbol{Retailer\ 1}}, \boldsymbol{Q}_{\boldsymbol{Retailer\ 3}})$

$$= \frac{1}{2}(-2c(\int_{\mathrm{q3}}^{\infty}(\int_{0}^{\mathrm{q1}}(\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}]\,d\mathrm{d1})\,d\mathrm{d3}$$

$$+\int_{0}^{\mathrm{q1}}(\int_{\mathrm{d1}}^{\mathrm{q3}}(\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}]\,d\mathrm{d3})\,d\mathrm{d1}$$

$$+\int_{\mathrm{q1}}^{\infty}(\int_{0}^{\mathrm{q3}}(\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d3}+\mathrm{q2}+\mathrm{q3}]\,d\mathrm{d3})\,d\mathrm{d1}$$

$$+\int_{0}^{\mathrm{q3}}(\int_{0}^{\mathrm{d3}}(\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d3}$$

$$+\int_{0}^{\mathrm{q1}}(\int_{0}^{\mathrm{d1}}(\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]\,d\mathrm{d3})\,d\mathrm{d1}$$

$$+\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}}(\int_{0}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}-\frac{1}{4}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+\mathrm{q2})]\,d\mathrm{d3}$$

$$+\frac{1}{2}\int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\mathrm{q1}}(-\mathrm{d1}+\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d2}$$

$$+\int_{\mathrm{q2}}^{\mathrm{q1+q2+q3}}(\frac{1}{2}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\mathrm{q3}}(-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+\mathrm{q2})]\,d\mathrm{d3}$$

$$+\int_{0}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}-\frac{1}{4}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d2}$$

$$+ \int_{0}^{q3} \left(\int_{d3}^{q1} (d1 - q1) f1[d1] f1[d3] f1[-2d3 + q2 + 2q3] \, dd1 \right) dd3$$

$$+ \int_{q3}^{\infty} \left(\int_{0}^{q1} ((-d1 + q1) f1[d1] f1[d3] f1[-d1 + q1 + q2]\right.$$

$$\left. + \int_{q2}^{-d1+q1+q2} -f1[d1] f1[d2] f1[d3] \, dd2) \, dd1 \right) dd3$$

$$+ \int_{q1}^{\infty} \left(\int_{0}^{q3} ((-d3 + q3) f1[d1] f1[d3] f1[-d3 + q2 + q3]\right.$$

$$\left. + \int_{q2}^{-d3+q2+q3} -f1[d1] f1[d2] f1[d3] \, dd2) \, dd3 \right) dd1$$

$$+ \int_{0}^{q1} \left(\int_{0}^{d1} (f1[d1] f1[d3] ((d1 - q1) f1[-2d1 + 2q1 + q2] + (-d3\right.$$

$$+ q3) f1[-d1 - d3 + q1 + q2 + q3])$$

$$\left. + \int_{-2d1+2q1+q2}^{-d1-d3+q1+q2+q3} -f1[d1] f1[d2] f1[d3] \, dd2) \, dd3 \right) dd1$$

$$+ \int_{0}^{q3} \left(\int_{0}^{d3} (f1[d1] f1[d3] ((-d1 + q1) f1[-d1 - d3 + q1 + q2 + q3] + (d3\right.$$

$$- q3) f1[-2d3 + q2 + 2q3])$$

$$\left. + \int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3} -f1[d1] f1[d2] f1[d3] \, dd2) \, dd1 \right) dd3$$

$$+\frac{1}{2}\left(\int_{q2}^{q1+q2+q3}\left(\frac{1}{2}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2\right.\right.$$

$$+q2)]\,dd3$$

$$+\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\left(\frac{1}{2}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]\right.$$

$$+\int_{0}^{\frac{1}{2}(-d2+q2)+q3}-f1[d1]f1[d2]f1[d3]\,dd3)\,dd1)\,dd2+(q1+q3)f1[q1+q2$$

$$+q3]\int_{0}^{\frac{q1-q3}{2}}\int_{0}^{\frac{1}{2}(-q1+q3)}f1[d1]f1[d3]\,dd3\,dd1)$$

$$+\frac{1}{2}\left(\int_{q2}^{q1+q2+q3}\left(\frac{1}{2}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)\right.\right.$$

$$+q3]\,dd1$$

$$+\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\left(\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]\right.$$

$$+\int_{0}^{q1+\frac{1}{2}(-d2+q2)}-f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2+(q1+q3)f1[q1+q2$$

$$+q3]\int_{0}^{\frac{1}{2}(-q1+q3)}\int_{0}^{\frac{q1-q3}{2}}f1[d1]f1[d3]\,dd1\,dd3)+f1[q1+q2$$

$$+q3]\int_{0}^{\frac{1}{2}(-q1+q3)}\int_{\frac{q1-q3}{2}}^{q1}(-d1+q1)f1[d1]f1[d3]\,dd1\,dd3+f1[q1+q2$$

$$+ \mathrm{q3}] \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \int_{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})}^{\mathrm{q3}} (-\mathrm{d3} + \mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}] \, d\mathrm{d3} \, d\mathrm{d1}) + (c$$

$$- \mathrm{tau})(2 \int_0^{\mathrm{q2}} (\int_0^{\mathrm{q3}} (-\mathrm{d2} + \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2} + \mathrm{q1} + \mathrm{q2}] \, d\mathrm{d3}) \, d\mathrm{d2}$$

$$+ 2 \int_0^{\mathrm{q2}} (\int_0^{\mathrm{q1}} (-\mathrm{d2} + \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[-\mathrm{d2} + \mathrm{q2} + \mathrm{q3}] \, d\mathrm{d1}) \, d\mathrm{d2}$$

$$+ 2 \int_0^{\mathrm{q2}} (\frac{1}{2} \int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2}+2\mathrm{q3})} (\mathrm{d3} - \mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})] \, d\mathrm{d3}$$

$$+ \int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (-\mathrm{d1} - \mathrm{d2} + \mathrm{q1} + \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[-\mathrm{d1} - \mathrm{d2} + \mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \, d\mathrm{d1}) \, d\mathrm{d2}$$

$$+ 2 \int_0^{\mathrm{q2}} (\int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (-\mathrm{d2} - \mathrm{d3} + \mathrm{q2} + \mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2} - \mathrm{d3} + \mathrm{q1}$$

$$+ \mathrm{q2} + \mathrm{q3}] \, d\mathrm{d3}$$

$$+ \frac{1}{2} \int_{\mathrm{q1}}^{\frac{1}{2}(-\mathrm{d2}+2\mathrm{q1}+\mathrm{q2})} (\mathrm{d1} - \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2} + \mathrm{q2}) + \mathrm{q3}] \, d\mathrm{d1}) \, d\mathrm{d2}$$

$$+ 2 \int_0^{\mathrm{q2}} (\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty} \frac{1}{4}(-\mathrm{d2} + \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})] \, d\mathrm{d3}$$

$$- \frac{1}{2} \int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (\mathrm{d1} - \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2} + \mathrm{q2}) + \mathrm{q3}] \, d\mathrm{d1}) \, d\mathrm{d2}$$

$$+ 2 \int_0^{\mathrm{q2}} (-\frac{1}{2} \int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (\mathrm{d3} - \mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})] \, d\mathrm{d3}$$

$$+\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\frac{1}{4}(-d2+q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]\,dd1)\,dd2$$

$$+2\int_0^{q2}(\int_0^{q3}((d2-q2)f1[d2]f1[d3]f1[-d2+q1+q2]$$

$$+\int_{-d2+q1+q2}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+\int_0^{q2}(-\frac{1}{2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]\,dd1$$

$$+\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]$$

$$+\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+2\int_0^{q2}(\frac{1}{2}\int_{\frac{1}{2}(-d2+2q1+q2)}^{\infty}-\frac{1}{2}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)$$

$$+q3]\,dd1$$

$$+\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}((d2+d3-q2-q3)f1[d2]f1[d3]f1[-d2-d3+q1+q2$$

$$+q3]+\int_{-d2-d3+q1+q2+q3}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+2\int_0^{q2}(\int_0^{q1}((d2-q2)f1[d1]f1[d2]f1[-d2+q2+q3]$$

$$+ \int_{-d2+q2+q3}^{\infty} f1[d1]f1[d2]f1[d3]\, dd3)\, dd1)\, dd2$$

$$+ 2\int_{0}^{q2} (\frac{1}{2}\int_{\frac{1}{2}(-d2+q2+2q3)}^{\infty} -\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2$$

$$+ q2)]\, dd3$$

$$+ \int_{q1}^{q1+\frac{1}{2}(-d2+q2)} ((d1+d2-q1-q2)f1[d1]f1[d2]f1[-d1-d2+q1+q2$$

$$+ q3] + \int_{-d1-d2+q1+q2+q3}^{\infty} f1[d1]f1[d2]f1[d3]\, dd3)\, dd1)\, dd2$$

$$+ \int_{0}^{q2} (-\frac{1}{2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty} (-d2+q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]\, dd3$$

$$+ \int_{q1+\frac{1}{2}(-d2+q2)}^{\infty} (\frac{1}{2}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]$$

$$+ \int_{\frac{1}{2}(-d2+q2)+q3}^{\infty} f1[d1]f1[d2]f1[d3]\, dd3)\, dd1)\, dd2$$

$$+ 2f1[q2]\int_{q3}^{q3}\int_{q1}^{-d3+q1+q3} (d1-q1)f1[d1]f1[d3]\, dd1\, dd3$$

$$+ 2f1[q2]\int_{q1}^{q1}\int_{-d1+q1+q3}^{\infty} (-d1+q1)f1[d1]f1[d3]\, dd3\, dd1$$

$$+ 2f1[q2]\int_{q1}^{q1}\int_{q3}^{-d1+q1+q3} (d3-q3)f1[d1]f1[d3]\, dd3\, dd1$$

$$+ 2\mathrm{f1}[\mathrm{q2}] \int_{\mathrm{q3}}^{\mathrm{q3}} \int_{-\mathrm{d3}+\mathrm{q1}+\mathrm{q3}}^{\infty} (-\mathrm{d3} + \mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}) - s(-2$$

$$+ 2 \int_{0}^{\mathrm{q2}} (\int_{0}^{\mathrm{q3}} (-\mathrm{d2} + \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2} + \mathrm{q1} + \mathrm{q2}]\, d\mathrm{d3})\, d\mathrm{d2}$$

$$+ 2 \int_{0}^{\mathrm{q2}} (\int_{0}^{\mathrm{q1}} (-\mathrm{d2} + \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[-\mathrm{d2} + \mathrm{q2} + \mathrm{q3}]\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+ 2 \int_{0}^{\mathrm{q2}} (\frac{1}{2} \int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2}+2\mathrm{q3})} (\mathrm{d3} - \mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})]\, d\mathrm{d3}$$

$$+ \int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (-\mathrm{d1} - \mathrm{d2} + \mathrm{q1} + \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[-\mathrm{d1} - \mathrm{d2} + \mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}]\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+ 2 \int_{0}^{\mathrm{q2}} (\int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (-\mathrm{d2} - \mathrm{d3} + \mathrm{q2} + \mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2} - \mathrm{d3} + \mathrm{q1}$$

$$+ \mathrm{q2} + \mathrm{q3}]\, d\mathrm{d3}$$

$$+ \frac{1}{2} \int_{\mathrm{q1}}^{\frac{1}{2}(-\mathrm{d2}+2\mathrm{q1}+\mathrm{q2})} (\mathrm{d1} - \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2} + \mathrm{q2}) + \mathrm{q3}]\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+ 2 \int_{0}^{\mathrm{q2}} (\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty} \frac{1}{4}(-\mathrm{d2} + \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})]\, d\mathrm{d3}$$

$$- \frac{1}{2} \int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (\mathrm{d1} - \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2} + \mathrm{q2}) + \mathrm{q3}]\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+ 2 \int_{0}^{\mathrm{q2}} (-\frac{1}{2} \int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (\mathrm{d3} - \mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})]\, d\mathrm{d3}$$

$$+\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\frac{1}{4}(-d2+q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]\,dd1)\,dd2$$

$$+2\int_0^{q2}(\int_0^{q3}((d2-q2)f1[d2]f1[d3]f1[-d2+q1+q2]$$

$$+\int_{-d2+q1+q2}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+\int_0^{q2}(-\frac{1}{2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2+q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]\,dd1$$

$$+\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]$$

$$+\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+2\int_0^{q2}(\frac{1}{2}\int_{\frac{1}{2}(-d2+2q1+q2)}^{\infty}-\frac{1}{2}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)$$

$$+q3]\,dd1$$

$$+\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}((d2+d3-q2-q3)f1[d2]f1[d3]f1[-d2-d3+q1+q2$$

$$+q3]+\int_{-d2-d3+q1+q2+q3}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+2\int_0^{q2}(\int_0^{q1}((d2-q2)f1[d1]f1[d2]f1[-d2+q2+q3]$$

$$+\int_{-\mathrm{d2}+\mathrm{q2}+\mathrm{q3}}^{\infty} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3})\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+2\int_{0}^{\mathrm{q2}} (\frac{1}{2}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2}+2\mathrm{q3})}^{\infty} -\frac{1}{2}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+\mathrm{q2})]\, d\mathrm{d3}$$

$$+\int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} ((\mathrm{d1}+\mathrm{d2}-\mathrm{q1}-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}$$

$$+\mathrm{q3}]+\int_{-\mathrm{d1}-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}^{\infty} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3})\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+\int_{0}^{\mathrm{q2}} (-\frac{1}{2}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty} (-\mathrm{d2}+\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})]\, d\mathrm{d3}$$

$$+\int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\infty} (\frac{1}{2}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}]$$

$$+\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3})\, d\mathrm{d1})\, d\mathrm{d2}$$

$$+2\mathrm{f1}[\mathrm{q2}]\int_{\mathrm{q3}}^{\mathrm{q3}}\int_{\mathrm{q1}}^{-\mathrm{d3}+\mathrm{q1}+\mathrm{q3}} (\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}$$

$$+2\mathrm{f1}[\mathrm{q2}]\int_{\mathrm{q1}}^{\mathrm{q1}}\int_{-\mathrm{d1}+\mathrm{q1}+\mathrm{q3}}^{\infty} (-\mathrm{d1}+\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+2\mathrm{f1}[\mathrm{q2}]\int_{\mathrm{q1}}^{\mathrm{q1}}\int_{\mathrm{q3}}^{-\mathrm{d1}+\mathrm{q1}+\mathrm{q3}} (\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ 2\mathrm{f1}[\mathrm{q2}]\int_{\mathrm{q3}}^{\mathrm{q3}}\int_{-\mathrm{d3}+\mathrm{q1}+\mathrm{q3}}^{\infty}(-\mathrm{d3}$$

$$+ \mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1}\,d\mathrm{d3})\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\mathrm{q2}}\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2}\,d\mathrm{d3}\,d\mathrm{d1}$$

$$- p(-2 - 2\int_{\mathrm{q3}}^{\infty}(\int_{0}^{\mathrm{q1}}(\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}]\,d\mathrm{d1})\,d\mathrm{d3}$$

$$- 2\int_{0}^{\mathrm{q1}}(\int_{\mathrm{d1}}^{\mathrm{q3}}(\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}]\,d\mathrm{d3})\,d\mathrm{d1}$$

$$- 2\int_{\mathrm{q1}}^{\infty}(\int_{0}^{\mathrm{q3}}(\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d3}+\mathrm{q2}+\mathrm{q3}]\,d\mathrm{d3})\,d\mathrm{d1}$$

$$- 2\int_{0}^{\mathrm{q3}}(\int_{0}^{\mathrm{d3}}(\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d3}$$

$$- 2\int_{0}^{\mathrm{q1}}(\int_{0}^{\mathrm{d1}}(\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]\,d\mathrm{d3})\,d\mathrm{d1}$$

$$- 2\int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}(\int_{0}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} -\frac{1}{4}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+ \mathrm{q2})]\,d\mathrm{d3}$$

$$+ \frac{1}{2}\int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\mathrm{q1}}(-\mathrm{d1}+\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d2}$$

$$- 2\int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}(\frac{1}{2}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\mathrm{q3}}(-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+ \mathrm{q2})]\,d\mathrm{d3}$$

$$+\int_0^{q1+\frac{1}{2}(-d2+q2)} -\frac{1}{4}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]\,dd1)\,dd2$$

$$-2\int_0^{q3}(\int_{d3}^{q1}(d1-q1)f1[d1]f1[d3]f1[-2d3+q2+2q3]\,dd1)\,dd3$$

$$-\int_{q2}^{q1+q2+q3}(\frac{1}{2}\int_0^{q1+\frac{1}{2}(-d2+q2)}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)$$

$$+q3]\,dd1$$

$$+\int_0^{\frac{1}{2}(-d2+q2)+q3}(\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]$$

$$+\int_0^{q1+\frac{1}{2}(-d2+q2)} -f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$-2\int_{q3}^{\infty}(\int_0^{q1}((-d1+q1)f1[d1]f1[d3]f1[-d1+q1+q2]$$

$$+\int_{q2}^{-d1+q1+q2} -f1[d1]f1[d2]f1[d3]\,dd2)\,dd1)\,dd3$$

$$-2\int_{q1}^{\infty}(\int_0^{q3}((-d3+q3)f1[d1]f1[d3]f1[-d3+q2+q3]$$

$$+\int_{q2}^{-d3+q2+q3} -f1[d1]f1[d2]f1[d3]\,dd2)\,dd3)\,dd1$$

$$-2\int_0^{q1}(\int_0^{d1}(f1[d1]f1[d3]((d1-q1)f1[-2d1+2q1+q2]+(-d3$$

$$+ \mathrm{q3})\mathrm{f1}[-\mathrm{d1} - \mathrm{d3} + \mathrm{q1} + \mathrm{q2} + \mathrm{q3}])$$

$$+ \int_{-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}}^{-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} -\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2})\, d\mathrm{d3})\, d\mathrm{d1}$$

$$- 2\int_0^{\mathrm{q3}} (\int_0^{\mathrm{d3}} (\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]((-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[-\mathrm{d1} - \mathrm{d3} + \mathrm{q1} + \mathrm{q2} + \mathrm{q3}] + (\mathrm{d3}$$

$$- \mathrm{q3})\mathrm{f1}[-2\mathrm{d3} + \mathrm{q2} + 2\mathrm{q3}])$$

$$+ \int_{-2\mathrm{d3}+\mathrm{q2}+2\mathrm{q3}}^{-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} -\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2})\, d\mathrm{d1})\, d\mathrm{d3}$$

$$- \int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} (\frac{1}{2}\int_0^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2}$$

$$+ \mathrm{q2})]\, d\mathrm{d3}$$

$$+ \int_0^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})} (\frac{1}{2}(\mathrm{d2} - \mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2} + \mathrm{q2}) + \mathrm{q3}]$$

$$+ \int_0^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} -\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3})\, d\mathrm{d1})\, d\mathrm{d2} - (\mathrm{q1} + \mathrm{q3})\mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3}\, d\mathrm{d1} - (\mathrm{q1} + \mathrm{q3})\mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3} - 2\mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \int_{\frac{\mathrm{q1}-\mathrm{q3}}{2}}^{\mathrm{q1}} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3} - 2\mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \int_{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})}^{\mathrm{q3}} (-\mathrm{d3}$$

$$+ \mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3}\, d\mathrm{d1}) \int_0^{\infty} \int_0^{\infty} \int_{\mathrm{q2}}^{\infty} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ 2r(\mathrm{q2} \int_0^{\infty} (\int_0^{\infty} -\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q2}]\, d\mathrm{d3})\, d\mathrm{d1}$$

$$+ \int_0^{\infty} (\int_0^{\infty} \mathrm{q2f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q2}]\, d\mathrm{d3})\, d\mathrm{d1}$$

$$+ \int_{\mathrm{q3}}^{\infty} (\int_0^{\mathrm{q1}} (\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}]\, d\mathrm{d1})\, d\mathrm{d3}$$

$$+ \int_0^{\mathrm{q1}} (\int_{\mathrm{d1}}^{\mathrm{q3}} (\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}]\, d\mathrm{d3})\, d\mathrm{d1}$$

$$+ \int_{\mathrm{q1}}^{\infty} (\int_0^{\mathrm{q3}} (\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d3}+\mathrm{q2}+\mathrm{q3}]\, d\mathrm{d3})\, d\mathrm{d1}$$

$$+ \int_0^{\mathrm{q3}} (\int_0^{\mathrm{d3}} (\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]\, d\mathrm{d1})\, d\mathrm{d3}$$

$$+ \int_0^{\mathrm{q1}} (\int_0^{\mathrm{d1}} (\mathrm{d3}-\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]\, d\mathrm{d3})\, d\mathrm{d1}$$

$$+ \int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} (\int_0^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} -\frac{1}{4}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+ \mathrm{q2})]\, d\mathrm{d3}$$

$$+\frac{1}{2}\int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\mathrm{q1}}(-\mathrm{d1}+\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d2}$$

$$+\int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}(\frac{1}{2}\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\mathrm{q3}}(-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}$$

$$+\mathrm{q2})]\,d\mathrm{d3}$$

$$+\int_{0}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}-\frac{1}{4}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d2}$$

$$+\int_{0}^{\mathrm{q3}}(\int_{\mathrm{d3}}^{\mathrm{q1}}(\mathrm{d1}-\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-2\mathrm{d3}+\mathrm{q2}+2\mathrm{q3}]\,d\mathrm{d1})\,d\mathrm{d3}$$

$$+\int_{\mathrm{q3}}^{\infty}(\int_{0}^{\mathrm{q1}}((-\mathrm{d1}+\mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}]$$

$$+\int_{\mathrm{q2}}^{-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}}-\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2})\,d\mathrm{d1})\,d\mathrm{d3}$$

$$+\int_{\mathrm{q1}}^{\infty}(\int_{0}^{\mathrm{q3}}((-\mathrm{d3}+\mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d3}+\mathrm{q2}+\mathrm{q3}]$$

$$+\int_{\mathrm{q2}}^{-\mathrm{d3}+\mathrm{q2}+\mathrm{q3}}-\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2})\,d\mathrm{d3})\,d\mathrm{d1}$$

$$+\int_{0}^{\mathrm{q1}}(\int_{0}^{\mathrm{d1}}(\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]((\mathrm{d1}-\mathrm{q1})\mathrm{f1}[-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}]+(-\mathrm{d3}$$

$$+\mathrm{q3})\mathrm{f1}[-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}])$$

$$+\int_{-2\mathrm{d1}+2\mathrm{q1}+\mathrm{q2}}^{-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}-\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d2})\,d\mathrm{d3})\,d\mathrm{d1}$$

$$+\int_0^{q3}(\int_0^{d3}(f1[d1]f1[d3]((-d1+q1)f1[-d1-d3+q1+q2+q3]+(d3$$

$$-q3)f1[-2d3+q2+2q3])$$

$$+\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}-f1[d1]f1[d2]f1[d3]\,dd2)\,dd1)\,dd3$$

$$+\frac{1}{2}(\int_{q2}^{q1+q2+q3}(\frac{1}{2}\int_0^{\frac{1}{2}(-d2+q2)+q3}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2$$

$$+q2)]\,dd3$$

$$+\int_0^{q1+\frac{1}{2}(-d2+q2)}(\frac{1}{2}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)+q3]$$

$$+\int_0^{\frac{1}{2}(-d2+q2)+q3}-f1[d1]f1[d2]f1[d3]\,dd3)\,dd1)\,dd2+(q1+q3)f1[q1+q2$$

$$+q3]\int_0^{\frac{q1-q3}{2}}\int_0^{\frac{1}{2}(-q1+q3)}f1[d1]f1[d3]\,dd3\,dd1)$$

$$+\frac{1}{2}(\int_{q2}^{q1+q2+q3}(\frac{1}{2}\int_0^{q1+\frac{1}{2}(-d2+q2)}(d2-q2)f1[d1]f1[d2]f1[\frac{1}{2}(-d2+q2)$$

$$+q3]\,dd1$$

$$+\int_0^{\frac{1}{2}(-d2+q2)+q3}(\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]$$

$$+\int_0^{q1+\frac{1}{2}(-d2+q2)}-f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2+(q1+q3)f1[q1+q2$$

$$+ \mathrm{q3}] \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}) + \mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \int_{\frac{\mathrm{q1}-\mathrm{q3}}{2}}^{\mathrm{q1}} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3} + \mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \int_{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})}^{\mathrm{q3}} (-\mathrm{d3} + \mathrm{q3})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ \int_0^{\infty} \int_0^{\infty} \int_{\mathrm{q2}}^{\infty} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1})$$

$$+ p(\int_0^{\infty} (\int_0^{\infty} -\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q2}]\, d\mathrm{d3})\, d\mathrm{d1})(2\mathrm{q2}$$

$$- 2 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \mathrm{d2}\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d3}\, d\mathrm{d1}$$

$$+ 2 \int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} \int_0^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} \int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\mathrm{q1}} (-\mathrm{d1}$$

$$+ \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}\, d\mathrm{d2}$$

$$+ 2 \int_0^{\mathrm{q3}} \int_0^{\mathrm{d3}} \int_{-\mathrm{d1}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$+ 2 \int_0^{\mathrm{q3}} \int_{\mathrm{d3}}^{\mathrm{q1}} \int_{-2\mathrm{d3}+\mathrm{q2}+2\mathrm{q3}}^{\infty} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2}\, d\mathrm{d1}\, d\mathrm{d3}$$

$$+ 2\int_{q3}^{\infty}\int_{0}^{q1}\int_{-d1+q1+q2}^{\infty}(-d1 + q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+ 2\int_{q1}^{\infty}\int_{0}^{q3}\int_{q2}^{-d3+q2+q3}(d2 - q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ \int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}(d2$$

$$- q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$+ \int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}(d2$$

$$- q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+ 2\int_{q3}^{\infty}\int_{0}^{q1}\int_{q2}^{-d1+q1+q2}(d2 - q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+ 2\int_{0}^{q1}\int_{0}^{d1}\int_{-2d1+2q1+q2}^{-d1-d3+q1+q2+q3}(d1 + d2 - q1$$

$$- q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2 + d3 - q2$$

$$- q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$+ 2\int_{0}^{q1}\int_{0}^{d1}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{0}^{q1}\int_{d1}^{q3}\int_{-2d1+2q1+q2}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{q1}^{\infty}\int_{0}^{q3}\int_{-d3+q2+q3}^{\infty}(-d3 + q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{q2}^{q1+q2+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}\int_{\frac{1}{2}(-d2+q2)+q3}^{q3}(-d3$$

$$+ q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2)$$

$$+ s(\int_{0}^{\infty}(\int_{0}^{\infty}f1[d1]f1[d3]f1[q2]\,dd3)\,dd1)(2q2$$

$$- 2\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}d2f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$- 2\int_{0}^{q2}\int_{0}^{q3}\int_{q1}^{-d2+q1+q2}(d1 - q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- 2\int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1$$

$$- q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- 2\int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1$$

$$- q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- 2\int_0^{q2}\int_0^{q1}\int_{-d2+q2+q3}^{\infty}(-d2 + q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$- \int_0^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(-d2$$

$$+ q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$- 2\int_0^{q2}\int_0^{q3}\int_{-d2+q1+q2}^{\infty}(-d2 + q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- \int_0^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2$$

$$+ q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- 2\int_0^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{-d1-d2+q1+q2+q3}^{\infty}(-d1 - d2 + q1$$

$$+ q2)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$- 2\int_0^{q2}\int_0^{q1}\int_{q3}^{-d2+q2+q3}(d3 - q3)f1[d1]f1[d2]f1[d3]\,dd3\,dd1\,dd2$$

$$- 2\int_{0}^{q2}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}\int_{q3}^{-d1-d2+q1+q2+q3}(d3$$

$$- q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d3\,\mathit{d}d1\,\mathit{d}d2$$

$$- 2\int_{0}^{q2}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d3$$

$$- q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d3\,\mathit{d}d1\,\mathit{d}d2$$

$$- 2\int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2 - d3 + q2$$

$$+ q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2))$$

Appendix H

Definition of $k_{nc,i}(Q_c, Q_{nc,i}, Q_{nc,j})$

The following equation is taken from a Wolfram Mathematica file. Here the central retailer is Retailer 2, and the non-central retailers are Retailer 1 and Retailer 3.

$k_{Retailer\ 1}(Q_{Retailer\ 2}, Q_{Retailer\ 1}, Q_{Retailer\ 3})$

$$= \frac{1}{2}(-c(2\int_0^{\mathrm{q2}}(\int_0^{\mathrm{q3}}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}]\,d\mathrm{d3})\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}(\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty}(\mathrm{d2}-\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})]\,d\mathrm{d3})\,d\mathrm{d2}$$

$$+2(\int_0^{\mathrm{q2}}(\int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}(\mathrm{d2}+\mathrm{d3}-\mathrm{q2}-\mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}$$

$$+\mathrm{q3}]\,d\mathrm{d3})\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}(\int_0^{\mathrm{q3}}((-\mathrm{d2}+\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}]$$

$$+\int_{\mathrm{q1}}^{-\mathrm{d2}+\mathrm{q1}+\mathrm{q2}}-\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1})\,d\mathrm{d3})\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}(\int_{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}^{\infty}(\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})]$$

$$+\int_{\mathrm{q1}}^{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}-\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1})\,d\mathrm{d3})\,d\mathrm{d2}$$

$$+\int_0^{\mathrm{q2}}(\int_{\mathrm{q3}}^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}}((-\mathrm{d2}-\mathrm{d3}+\mathrm{q2}+\mathrm{q3})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d2}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}]$$

$$+\int_{\mathrm{q1}}^{-\mathrm{d2}-\mathrm{d3}+\mathrm{q1}+\mathrm{q2}+\mathrm{q3}}-\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\,d\mathrm{d1})\,d\mathrm{d3})\,d\mathrm{d2}))+2(c$$

$$- \mathrm{tau})\left(\int_{q3}^{\infty}\left(\int_{0}^{q1}(-d1+q1)f1[d1]f1[d3]f1[-d1+q1+q2]\,dd1\right)dd3\right.$$

$$+\int_{0}^{q3}\left(\int_{0}^{d3}(-d1+q1)f1[d1]f1[d3]f1[-d1-d3+q1+q2+q3]\,dd1\right)dd3$$

$$+\int_{q2}^{q1+q2+q3}\left(\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\left(-\frac{1}{2}(d2-q2)f1[d2]f1[d3]f1\left[q1+\frac{1}{2}(-d2+q2)\right]\right.\right.$$

$$\left.\left.+\int_{q1+\frac{1}{2}(-d2+q2)}^{q1}f1[d1]f1[d2]f1[d3]\,dd1\right)dd3\right)dd2$$

$$+\int_{q3}^{\infty}\left(\int_{0}^{q1}\left((d1-q1)f1[d1]f1[d3]f1[-d1+q1+q2]\right.\right.$$

$$\left.\left.+\int_{-d1+q1+q2}^{\infty}f1[d1]f1[d2]f1[d3]\,dd2\right)dd1\right)dd3$$

$$+\int_{0}^{q3}\left(\int_{0}^{d3}\left((d1-q1)f1[d1]f1[d3]f1[-d1-d3+q1+q2+q3]\right.\right.$$

$$\left.\left.+\int_{-d1-d3+q1+q2+q3}^{\infty}f1[d1]f1[d2]f1[d3]\,dd2\right)dd1\right)dd3$$

$$+\int_{0}^{q3}\left(\int_{d3}^{q1}\left(\int_{-2d3+q2+2q3}^{\infty}f1[d1]f1[d2]f1[d3]\,dd2\right)dd1\right)dd3$$

$$+\frac{1}{2}\left(\int_{q2}^{q1+q2+q3}\left(\int_{0}^{\frac{1}{2}(-d2+q2)+q3}(d2-q2)f1[d2]f1[d3]f1\left[q1+\frac{1}{2}(-d2+q2)\right]dd3\right)dd2\right.$$

$$+ (\mathrm{q1} + \mathrm{q3})\mathrm{f1}[\mathrm{q1} + \mathrm{q2} + \mathrm{q3}] \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \int_0^{\frac{\mathrm{q1}-\mathrm{q3}}{2}} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}) + \mathrm{f1}[\mathrm{q1} + \mathrm{q2}$$

$$+ \mathrm{q3}] \int_0^{\frac{1}{2}(-\mathrm{q1}+\mathrm{q3})} \int_{\frac{\mathrm{q1}-\mathrm{q3}}{2}}^{\mathrm{q1}} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1}\, d\mathrm{d3}) + s(2$$

$$- 2 \int_{\mathrm{q3}}^{\infty} (\int_0^{\mathrm{q1}} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1} + \mathrm{q1} + \mathrm{q2}]\, d\mathrm{d1})\, d\mathrm{d3}$$

$$- \int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} (\int_0^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (\mathrm{d2} - \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})]\, d\mathrm{d3})\, d\mathrm{d2}$$

$$- 2 \int_0^{\mathrm{q3}} (\int_0^{\mathrm{d3}} (-\mathrm{d1} + \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1} - \mathrm{d3} + \mathrm{q1} + \mathrm{q2} + \mathrm{q3}]\, d\mathrm{d1})\, d\mathrm{d3}$$

$$- 2 \int_{\mathrm{q2}}^{\mathrm{q1}+\mathrm{q2}+\mathrm{q3}} (\int_0^{\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})+\mathrm{q3}} (-\frac{1}{2}(\mathrm{d2} - \mathrm{q2})\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[\mathrm{q1} + \frac{1}{2}(-\mathrm{d2} + \mathrm{q2})]$$

$$+ \int_{\mathrm{q1}+\frac{1}{2}(-\mathrm{d2}+\mathrm{q2})}^{\mathrm{q1}} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d1})\, d\mathrm{d3})\, d\mathrm{d2}$$

$$- 2 \int_{\mathrm{q3}}^{\infty} (\int_0^{\mathrm{q1}} ((\mathrm{d1} - \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1} + \mathrm{q1} + \mathrm{q2}]$$

$$+ \int_{-\mathrm{d1}+\mathrm{q1}+\mathrm{q2}}^{\infty} \mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d2}]\mathrm{f1}[\mathrm{d3}]\, d\mathrm{d2})\, d\mathrm{d1})\, d\mathrm{d3}$$

$$- 2 \int_0^{\mathrm{q3}} (\int_0^{\mathrm{d3}} ((\mathrm{d1} - \mathrm{q1})\mathrm{f1}[\mathrm{d1}]\mathrm{f1}[\mathrm{d3}]\mathrm{f1}[-\mathrm{d1} - \mathrm{d3} + \mathrm{q1} + \mathrm{q2} + \mathrm{q3}]$$

$$+\int_{-d1-d3+q1+q2+q3}^{\infty} f1[d1]f1[d2]f1[d3]\,dd2)\,dd1)\,dd3$$

$$-2\int_0^{q3}(\int_{d3}^{q1}(\int_{-2d3+q2+2q3}^{\infty} f1[d1]f1[d2]f1[d3]\,dd2)\,dd1)\,dd3-(q1+q3)f1[q1+q2$$

$$+q3]\int_0^{\frac{1}{2}(-q1+q3)}\int_0^{\frac{q1-q3}{2}} f1[d1]f1[d3]\,dd1\,dd3-2f1[q1+q2$$

$$+q3]\int_0^{\frac{1}{2}(-q1+q3)}\int_{\frac{q1-q3}{2}}^{q1}(-d1$$

$$+q1)f1[d1]f1[d3]\,dd1\,dd3)\int_0^{\infty}\int_0^{\infty}\int_0^{q1} f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+p(2\int_0^{q2}(\int_0^{q3}(d2-q2)f1[d2]f1[d3]f1[-d2+q1+q2]\,dd3)\,dd2$$

$$+\int_0^{q2}(\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]\,dd3)\,dd2+2(1$$

$$+\int_0^{q2}(\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d2+d3-q2-q3)f1[d2]f1[d3]f1[-d2-d3+q1+q2$$

$$+q3]\,dd3)\,dd2$$

$$+\int_0^{q2}(\int_0^{q3}((-d2+q2)f1[d2]f1[d3]f1[-d2+q1+q2]$$

$$+\int_{q1}^{-d2+q1+q2}-f1[d1]f1[d2]f1[d3]\,dd1)\,dd3)\,dd2$$

$$+\int_0^{q2}\left(\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\left(\frac{1}{2}(-d2+q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]\right.\right.$$

$$\left.\left.+\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}-f1[d1]f1[d2]f1[d3]\,dd1\right)dd3\right)dd2$$

$$+\int_0^{q2}\left(\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\left((-d2-d3+q2+q3)f1[d2]f1[d3]f1[-d2-d3+q1+q2+q3]\right.\right.$$

$$\left.\left.\left.+\int_{q1}^{-d2-d3+q1+q2+q3}-f1[d1]f1[d2]f1[d3]\,dd1\right)dd3\right)dd2\right)\right)\int_0^{\infty}\int_0^{\infty}\int_{q1}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+2r\left(q1\int_0^{\infty}\left(\int_0^{\infty}-f1[d2]f1[d3]f1[q1]\,dd3\right)dd2+\int_0^{\infty}\left(\int_0^{\infty}q1f1[d2]f1[d3]f1[q1]\,dd3\right)dd2\right.$$

$$+\int_0^{q2}\left(\int_0^{q3}(d2-q2)f1[d2]f1[d3]f1[-d2+q1+q2]\,dd3\right)dd2$$

$$+\frac{1}{2}\int_0^{q2}\left(\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}(d2-q2)f1[d2]f1[d3]f1[q1+\frac{1}{2}(-d2+q2)]\,dd3\right)dd2$$

$$+\int_0^{q2}\left(\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}(d2+d3-q2-q3)f1[d2]f1[d3]f1[-d2-d3+q1+q2\right.$$

$$\left.+q3]\,dd3\right)dd2$$

$$+\int_0^{q2}\left(\int_0^{q3}\left((-d2+q2)f1[d2]f1[d3]f1[-d2+q1+q2]\right.\right.$$

$$\left.\left.+\int_{q1}^{-d2+q1+q2}-f1[d1]f1[d2]f1[d3]\,dd1\right)dd3\right)dd2$$

$$+\int_{0}^{q2}\left(\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\left(\frac{1}{2}(-d2+q2)f1[d2]f1[d3]f1\left[q1+\frac{1}{2}(-d2+q2)\right]\right.\right.$$

$$\left.\left.+\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}-f1[d1]f1[d2]f1[d3]\,dd1\right)dd3\right)dd2$$

$$+\int_{0}^{q2}\left(\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\left((-d2-d3+q2+q3)f1[d2]f1[d3]f1[-d2-d3+q1\right.\right.$$

$$\left.\left.+q2+q3]+\int_{q1}^{-d2-d3+q1+q2+q3}-f1[d1]f1[d2]f1[d3]\,dd1\right)dd3\right)dd2$$

$$+\int_{0}^{\infty}\int_{0}^{\infty}\int_{q1}^{\infty}f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2)$$

$$+s\left(\int_{0}^{\infty}\left(\int_{0}^{\infty}f1[d2]f1[d3]f1[q1]\,dd3\right)dd2\right)(2q1$$

$$-2\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}d1f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$-2\int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1+\frac{1}{2}(-d2+q2)}^{q1}(-d1$$

$$+q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$-2\int_{0}^{q3}\int_{0}^{d3}\int_{-d1-d3+q1+q2+q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$- 2\int_{0}^{q3}\int_{d3}^{q1}\int_{-2d3+q2+2q3}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$- 2\int_{q3}^{\infty}\int_{0}^{q1}\int_{-d1+q1+q2}^{\infty}(-d1+q1)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$- \int_{q2}^{q1+q2+q3}\int_{0}^{\frac{1}{2}(-d2+q2)+q3}\int_{0}^{q1+\frac{1}{2}(-d2+q2)}(d2$$

$$- q2)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$- 2\int_{q3}^{\infty}\int_{0}^{q1}\int_{q2}^{-d1+q1+q2}(d2-q2)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3$$

$$- 2\int_{0}^{q3}\int_{0}^{d3}\int_{-2d3+q2+2q3}^{-d1-d3+q1+q2+q3}(d2+d3-q2$$

$$- q3)f1[d1]f1[d2]f1[d3]\,dd2\,dd1\,dd3)$$

$$+ p(\int_{0}^{\infty}(\int_{0}^{\infty}-f1[d2]f1[d3]f1[q1]\,dd3)\,dd2)(2q1$$

$$- 2\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}d1f1[d1]f1[d2]f1[d3]\,dd2\,dd3\,dd1$$

$$+ 2\int_{0}^{q2}\int_{0}^{q3}\int_{q1}^{-d2+q1+q2}(d1-q1)f1[d1]f1[d2]f1[d3]\,dd1\,dd3\,dd2$$

$$+ 2\int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{q1}^{-d2-d3+q1+q2+q3}(d1 - q1)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2$$

$$+ 2\int_{0}^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1}^{q1+\frac{1}{2}(-d2+q2)}(d1 - q1)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2$$

$$+ 2\int_{0}^{q2}\int_{0}^{q3}\int_{-d2+q1+q2}^{\infty}(-d2 + q2)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2$$

$$+ \int_{0}^{q2}\int_{\frac{1}{2}(-d2+q2)+q3}^{\infty}\int_{q1+\frac{1}{2}(-d2+q2)}^{\infty}(-d2 + q2)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2$$

$$+ 2\int_{0}^{q2}\int_{q3}^{\frac{1}{2}(-d2+q2)+q3}\int_{-d2-d3+q1+q2+q3}^{\infty}(-d2 - d3 + q2 + q3)f1[d1]f1[d2]f1[d3]\,\mathit{d}d1\,\mathit{d}d3\,\mathit{d}d2))$$